

# Two formulas of planetary motion

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## Kepler's first law

The first Kepler's law might be expressed via a formula, expressing the distance  $r$  between the Sun and its orbiting planet as a function of the true anomaly  $\theta$ :

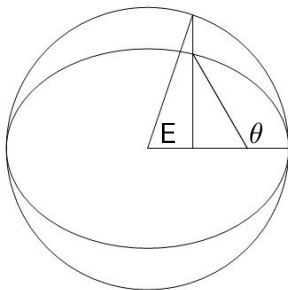
$$r = r(\theta) = \frac{p}{1 + e \cos \theta}, \quad p := \frac{b^2}{a} = a(1 - e^2),$$

where  $p$  is the (so-called) semilatus rectum, whereas  $a$  and  $b$  are the lengths of semi-major and semi-minor axes, respectively. The ellipticity of the orbit imposes upon its eccentricity  $e$  the condition:  $-1 < e < 1$ .

# Kepler's equation

Kepler's equation calculates the (so-called) mean anomaly  $M$  via the eccentric anomaly  $E$  as

$$M = E - e \sin E.^1$$



Eccentric and true anomalies

We might, as well, calculate the mean anomaly  $M$  as a function of the true anomaly  $\theta$  as

$$M = M(\theta) = 2 \arctan \left( \sqrt{\frac{1-e}{1+e}} \tan \left( \frac{\theta}{2} \right) \right) - \frac{e \sqrt{1-e^2} \sin \theta}{1+e \cos \theta},$$

<sup>1</sup>At the end of the Fourth Part of his work "De Motibus Stellae Martis", Kepler states, according to a translation from the Latin [1], concerning the solution of the problem so long known by his name (that is, concerning the determination of  $E$  for a given  $M$ ): "I am sufficiently satisfied that it cannot be solved a priori, on account of the different nature of the arc and the sine. But if I am mistaken, and any one shall point out the way to me, he will be in my eyes the great Apollonius."

## A formula unifying Kepler's second and third laws

One might verify the equivalence of the two formulas (for the mean anomaly  $M$ ) via the accessory identities:

$$\cos \theta = \frac{\cos E - e}{1 - e \cos E}, \quad \sin \theta = \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E}, \quad \tan \left( \frac{\theta}{2} \right) = \sqrt{\frac{1 + e}{1 - e}} \tan \left( \frac{E}{2} \right).$$

Now, the second and the third Kepler's laws might be unified (and strengthened) via a formula for the orbiting time  $t = t(\theta)$  as

$$t = \sqrt{\frac{a^3}{\mu}} M, \quad (1)$$

where  $\mu$  is the product of the mass of the Sun with the gravitational constant.

## Kepler's second and third laws (united and strengthened)

Differentiating the latter time function  $t$  with respect to the true anomaly  $\theta$  and taking the reciprocal, we readily calculate  $\dot{\theta}$ , thereby deriving a strong version of Kepler's second law as

$$\dot{\theta} r^2 = \sqrt{\mu p},$$

where the dot above denotes differentiation with respect to time.

Furthermore, integrating the latter equality, over a full period  $T$ , readily yields Kepler's third law as

$$2\pi ab = \sqrt{\mu p} T \implies T = 2\pi \sqrt{\frac{a^3}{\mu}}.$$

## The lengths of the four seasons

Moreover, having explicitly expressed  $t$  as a function of  $\theta$ , we might calculate the lengths of the four seasons on Earth. Assuming  $e = 5/299$ ,  $T = 1461/4$  and the value of the true anomaly  $\theta$  at the vernal equinox is  $3\pi/7$ , we calculate the lengths of Winter and Spring as

$$t\left(\frac{3\pi}{7}\right) - t\left(-\frac{\pi}{14}\right) \approx 88.995, \quad t\left(\frac{13\pi}{14}\right) - t\left(\frac{3\pi}{7}\right) \approx 92.765,$$

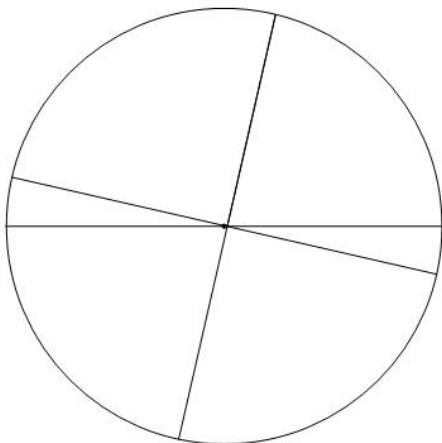
respectively. If we maintain the values of the argument  $\theta$  but flip the sign of eccentricity ( $e \mapsto -e$ ), then the two differences (upon evaluating  $t$ ) would correspond to the lengths of Summer (93.651) and Autumn (89.839), respectively. Thus, the length of the “polar night” at the North (South) Pole is 178.83 (186.42).<sup>2</sup>

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<sup>2</sup>The Winter at the North Pole does not get nearly as cold as the Summer gets at the South Pole, as consistently recorded and observed. The length of the “polar night” at the North Pole is over a week shorter than its length at the South Pole (although the small eccentricity of Earth's orbit makes it visibly indistinguishably from a circle). No temperature lower than that recorded at Mount McKinley, Alaska ( $-73.8^\circ\text{C}$ ) was ever recorded at the Northern Hemisphere of Earth. Yet, the lowest temperature ever directly recorded at ground level on Earth is  $-89.2^\circ\text{C}$  (184.0 K), which occurred at the Soviet Vostok Station in Antarctica on July 21, 1983.

## Causes of discrepancies

These calculations closely agree with the actual lengths of seasons, which are subject to small alterations which order of magnitude matches the (20 minutes) discrepancy between the tropical and the sidereal year along with other lesser components of precession (and nutation) of the equinoxes, such as the Chandler wobble [2].



# The rotating celestial sphere

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# References

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