The Gauss-Euler Conjugate Algorithm

A Quadratically Convergent, Parallelizable Algorithm for the Reciprocal of π

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The Gauss–Euler Algorithm is monotonically, quadratically convergent to π:

$$a_0 = \frac{1}{\sqrt{2}}, \qquad b_0 = 1, \qquad t_0 = \frac{1}{4}.$$

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n},\ t_{n+1} = t_n - 2^n (a_{n+1} - a_n)^2,$$

$$\frac{a_{n+1}^2}{t_n} \xrightarrow[n\to\infty]{} \pi.$$

Although it requires only three state variables, they are mutually coupled and so the Gauss–Euler algorithm is not parallelizable.

The naming of the algorithm credits its two most significant contributors [1,2].[1] Here we aim to present a Gauss–Euler Conjugate Algorithm, requiring only two variables to trace, yet it is parallelizable and monotonically, quadratically convergent to $1/\pi$:

$$\alpha_0 = \frac{1}{\sqrt{2}}, \qquad s_0 = \sqrt{\sqrt{2}},$$

$$T_n = s_n + \frac{1}{s_n},\ \alpha_{n+1} = \frac{1+\alpha_n}{T_n},\ s_{n+1} = \sqrt{\frac{T_n}{2}},$$

$$2^n(1-\alpha_n) \xrightarrow[n\to\infty]{} \frac{1}{\pi}.$$

The Gauss–Euler Algorithm converges to π ascendingly whereas the Conjugate Algorithm converges to $1/\pi$ ascendingly and so it converges to π descendingly.[2]

Note that the recurrence for s_n:

$$s_{n+1} = \sqrt{\frac{s_n + 1/s_n}{2}}$$

is entirely independent of the recurrence for α_n. This decoupling allows the following pipelined execution:

- **Core 1:** Given s_n, computes $1/s_n$, then T_n, then s_{n+1}.
- **Core 2:** Receives T_n from Core 1 and computes α_{n+1}.

[1] An objection raised in [3] was refuted in [4].

[2] We shall occasionally, for the sake of brevity, say "the Conjugate Algorithm" without implying any lesser credit to either Gauss or Euler.

In a pipelined implementation, the wall-clock time per iteration is dominated by the slower of the two cores — Core 1 (which handles the square root), while Core 2 remains idle after its quick division. In the Gauss–Euler Algorithm, by contrast, the updates for a_n, b_n, and t_n are mutually coupled, forcing sequential execution and precluding parallelization. Thus, though the Conjugate Algorithm requires two variable divisions per iteration (against none in a division-optimised Gauss–Euler Algorithm), the overlap of work on two cores can make their wall-clock times comparable. Running the two algorithms side by side is therefore worth considering for record-breaking π computation.

Convergence of the Gauss-Euler Algorithm and its Conjugate Algorithm to π

Iteration	Gauss-Euler Algorithm	Conjugate Algorithm
0	2.91421356237309504880 16	**3**.41421356237309504880 16
1	**3.14**057925052216824831 13	**3.14**260675394162260079 07
2	**3.1415926**4621354228214 93	**3.1415926**6096604423049 77
3	**3.14159265358979323 8**2795	**3.14159265358979323 8**6457

Both algorithms might be viewed as instances of the Legendre relation which was given in [1,2] as

$$\pi = \frac{2M(\beta)M(\gamma)}{N(\beta^2) + N(\gamma^2) - 1},$$

where $M(x)$ is the arithmetic-geometric mean of 1 and x and $N(x)$ is the modified arithmetic-geometric mean of 1 and x, whereas β and γ are assumed to be two positive numbers whose squares sum to one: $\beta^2 + \gamma^2 = 1$.

Nevertheless, the Legendre relation may be so interpreted so as to extend its domain to variables, not necessarily real. In particular,

$$\pi = \frac{M\left(\sqrt{2}\right)^2}{N(2) - 1} = \frac{2M\left(\sqrt{-1}\right)^2}{N(-1)}.$$

The Gauss-Euler Algorithm might thus be regarded as corresponding to the value $\beta = \sqrt{2}$, whereas the Conjugate Algorithm is then recognized as corresponding to the "conjugate" value $i = \sqrt{1 - \beta^2}$, once we notice that it could have been initiated with the complex values $\alpha_{-1} = 0$ and $s_{-1} = \sqrt{i}$, merging with the described "real-valued" Conjugate Algorithm after a single iteration!

1. Semjon Adlaj, "An Eloquent Formula for the Perimeter of an Ellipse," *Notices of the American Mathematical Society*, Vol. 59, No. 8, September 2012, pp. 1094–1099. `https://www.ams.org/notices/201208/rtx120801094p.pdf`

2. Semjon Adlaj, "Guan yu tuo yuan zhou chang de yi ge wan mei de ji suan gong shi" (Chinese translation of [1]), Yao Jingqi (transl.), Zhao Chunlai (rev.), *Shu Xue Yi Lin (Mathematics in Translation)*, Vol. 32, No. 1, 2013, pp. 30–37. `https://SemjonAdlaj.com/SP/13-1-Step3.pdf`

3. Richard Brent, "Old and New Algorithms for π," *Notices of the American Mathematical Society*, Vol. 60, No. 1, January 2013, p. 7. `https://www.ams.org/notices/201301/rnoti-p7.pdf`

4. ChatGPT (OpenAI, GPT-5.5), "On the Designation "Gauss–Euler Algorithm," July 2026. `https://SemjonAdlaj.com/SP/AIanswersBrent.pdf`