An analytic unifying formula of oscillatory and rotary motion of a simple pendulum

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The motion of a simple pendulum, whether oscillatory or rotary, is explicitly expressed as an analytic elliptic function of time, which lattice is either rhombic, in the first (oscillatory) case, or rectangular, in the second (rotary) case. The corresponding period, whether it is the time required for a swing (in the oscillatory case) or for a revolution (in the rotary case), is most efficiently calculated via Gauss arithmetic-geometric mean method (for evaluating complete elliptic integrals of the first kind). The doubly periodic solution "degenerates" to a singly periodic (equilibrium) solution, which period is either real (for stable equilibrium) or imaginary (for unstable equilibrium). Without the latter (elementary) solution, being a critical instance of an (all motion modes inclusive) analytic unifying solution, this fundamental problem of classical mechanics had never been entirely solved!

1 Introduction

Calculating the period of a simple pendulum in many "popular" references on elliptic functions, such as [10, p. 59, 77], and "authoritative" references on mechanics, such as [12, p. 73], have relied on a "classical" presentation of the motion of the pendulum, given in [7]. The latter reference included a chapter on Landen transformations. Yet, its leading author Appell was unaware of Gauss most efficient method for calculating complete elliptic integrals of the first kind, outlined in [3, 4], and although he did attain a mechanical interpretation of the imaginary period [6], he fell short of finding an analytic expression for the period, convergent for all values of the maximal angle, including complex(!) values, corresponding to rotary modes of motion. The ubiquitous use of power series for calculating the period (after Appell) has led to restricting these calculations to "small angles" along with introducing (superfluous) terms such as "the circular error", measuring the deviation of the period (with a given maximal angle) from the corresponding period of oscillation with an infinitely small maximal angle. Some authors have even (ridiculously) alleged that no analytic formulas for arbitrary maximal angles exist. Not until 2013(!) did that (exact and most efficient) formula (which we shall fully expose in all its grace) make its début to a textbook, namely [11, Appendix O: The Simple Plane Pendulum: Exact Solution], thereby filling a major gap (which had persisted for over two centuries!)[1] in all textbooks on mechanics. In fact, and moreover, the motion of the pendulum in either oscillatory or rotary mode might (and must) be represented by an analytic function of time, dependent on a single parameter which values lie either on the unit circle (centred at the origin in the complex plane) in the first (oscillatory) case, or on the real line in the second (rotary) case. The two special values ± 1, being at the intersection, would then represent the two (stable and unstable) equilibria states of the pendulum.

2 An essential elliptic function

Given a complex parameter β, introduce an *essential elliptic function* $\mathcal{R} = \mathcal{R}(t) = \mathcal{R}(t, \beta)$, as in [1, 2, 3, 4, 5], that is a (meromorphic) function $\mathcal{R}$, possessing a (double) pole at the origin and satisfying the differential equation

$$\dot{\mathcal{R}}^2 = 4\,\mathcal{R}\,(\mathcal{R} + \beta)\,(\mathcal{R} + 1/\beta)\,,$$

with the dot above denoting differentiation with respect to the first argument (t).
Let $\mathbb{H}$ denote the upper half of the complex plane $\mathbb{C}$, that is

$$\mathbb{H} = \{z \in \mathbb{C} : \ \Im(z) > 0\}.$$

[1]The discovery of most efficient calculation (via the arithmetic-geometric mean) of complete elliptic integrals of the first kind was reported by Gauss in his diary on May 30, 1799, and brought into blossom on December 16, 2011 when *the modified arithmetic-geometric mean*, for calculating complete elliptic integrals of the second kind, was introduced. An overview of this story is given in "An eloquent formula for the perimeter of an ellipse", Notices of the AMS **59**(8) (2012): 1094–1099.

Denote by $\mathbb{D}$ the unit disk (centred at the origin) in the complex plane, and let $\mathbb{T}$ denote its boundary (which we shall refer to as the unit circle), that is

$$\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}, \ \mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : \ |z| = 1\}.$$

Attach the subscript "+" in order to indicate an intersection (of a set) with $\mathbb{H}$, that is

$$\mathbb{D}_+ := \mathbb{D} \cap \mathbb{H}, \ \mathbb{T}_+ := \mathbb{T} \cap \mathbb{H},$$

and attach the subscript "−" in order to indicate the reflection of such an intersection across the real axis, that is

$$\mathbb{D}_- := \mathbb{D}\backslash\overline{\mathbb{D}}_+, \ \mathbb{T}_- := \mathbb{T}\backslash\overline{\mathbb{T}}_+,$$

where we have used the bar above a set to denote its topological closure.
Observe that restricting the value of β to lie in the union $\mathbb{D}_- \cup \overline{\mathbb{D}}_+$ would not cause any exclusion of members of the (parametric) family of essential elliptic functions. Observe, furthermore, that the doubly periodic function $\mathcal{R}$ degenerates to a (singly) periodic function as β approaches either 0 or ± 1. Denoting $\sqrt{-1}$ by i, we might explicitly write

$$\lim_{\beta\to 0} \beta\,\mathcal{R}(\sqrt{\beta}\,t, \, -\beta) = (\csc t)^2 := -\left(\frac{2}{e^{it} - e^{-it}}\right)^2,$$

$$\lim_{\beta\to 0} \beta\,\mathcal{R}(\sqrt{\beta}\,t, \, \beta) = (\operatorname{csch} t)^2 := \left(\frac{2}{e^{t} - e^{-t}}\right)^2,$$

$$\mathcal{R}(t, \, 1) = (\cot t)^2 := -\left(\frac{e^{2it}+1}{e^{2it}-1}\right)^2, \ \mathcal{R}(t, \, -1) = (\coth t)^2 := \left(\frac{e^{2t}+1}{e^{2t}-1}\right)^2.$$

Moreover, the functional relation

$$\mathcal{R}(i\,t, \, \beta) = -\mathcal{R}(t, \, -\beta) \tag{1}$$

holds $\forall\,\beta \in \mathbb{C}$.
Let $\Lambda = \Lambda(\beta)$ denote the period lattice of the elliptic function $\mathcal{R}$, and let K denote the quotient of the complex plane $\mathbb{C}$ modulo the lattice Λ, that is $K := \mathbb{C}\,/\Lambda$. Introduce a formal expression $I = I(\beta, \gamma)$ for the elliptic integral (of the first kind)

$$I(\beta, \gamma) := \int_0^\gamma \frac{dz}{\sqrt{z(z+\beta)(z+1/\beta)}}.$$

The integral I can not, of course, be evaluated independently of the path of integration chosen, yet its value is well defined, if regarded as an element of the torus $2K$, where K has just been defined. With this interpretation of I, endowing its formal expression with a concrete evaluation, we proceed to presenting a pair of values of (complete) elliptic integrals

$$w(\beta) := I(\beta, -\beta), \ u(\beta) := I(\beta, 1),$$

which, in fact, are most efficiently calculated via Gauss method, presented in [3, 4]:

$$w(\beta) = \pi\sqrt{-\beta}/M\left(\sqrt{1-\beta^2}\right), \ 2\,u(\beta) = \pi\sqrt{\beta}/M\left(\beta\right), \tag{2}$$

where $M(x)$ is the arithmetic-geometric mean of 1 and x. Note that M is not single valued, and although its argument is defined only up to a sign (being a two-valued square root) in the first of the latter equalities (2), both (equalities) are well defined (up to twice a period) even if the sign of either argument (of M) is flipped, as further explained in [5].
For a fixed β, set *the periods* $w_\pm$ and *the half-periods* $u_\pm$, of the elliptic function $\mathcal{R}(\cdot) = \mathcal{R}(\cdot, \beta)$, as

$$w_+ := w(\beta), \ w_- := w\left(1/\beta\right), \ u_+ := u(\beta), \ u_- := iu(-\beta).$$

The labels "periods" and "half-periods" assigned to $w_\pm$ and to $u_\pm$, respectively, are justified since

$$\mathcal{R}(w_\pm) = \mathcal{R}(2u_\pm) = \infty, \ \mathcal{R}(u_\pm) = 0,$$

and, furthermore,

$$\mathcal{R}\left(\frac{w_+}{2}\right) = \frac{-1}{\beta}, \ \mathcal{R}\left(\frac{w_-}{2}\right) = -\beta, \ \mathcal{R}\left(\frac{u_+}{2}\right) = 1, \ \mathcal{R}\left(\frac{u_-}{2}\right) = -1.$$

One might, as well, observe the relations

$$w_+ = u_+ - u_-,\ w_- = u_+ + u_-,^{2}$$

as well as, that the lattice $\Lambda = \mathbb{Z}w_+ \oplus \mathbb{Z}w_-$ is a sublattice of the lattice $\Lambda_2 := \mathbb{Z}u_+ \oplus \mathbb{Z}u_- = \mathbb{Z}w_\pm \oplus \mathbb{Z}u_\pm$ of index 2, and that the lattice $2\Lambda_2$ is, in its turn, a sublattice of the lattice Λ of index 2.
Before we move on to discussing the motion of a simple pendulum note that the functional relation (1) stems from the lattice relation

$$\Lambda(-\beta) = i\,\Lambda(\beta).$$

Allow β to assume arbitrary complex value aside from the triple $\{0, \pm 1\}$, so we might assume that $\beta = e^{i\phi}$, where ϕ is not a multiple of π, and let Λ_2 be the lattice defined in the preceding paragraph, so that the lattice Λ is a sublattice of the lattice Λ_2 of index 2, and $2\Lambda_2$ (being a sublattice of index 2 in Λ) is expressable as

$$2\Lambda_2 = 2\left(\mathbb{Z}u_+ \oplus \mathbb{Z}u_-\right) =$$

$$= \pi e^{i\phi_2}\left(\frac{\mathbb{Z}}{M\left(e^{i\phi}\right)} \oplus \frac{\mathbb{Z}}{M\left(-e^{i\phi}\right)}\right) = \pi\left(\frac{\mathbb{Z}}{M\left(\cos\phi_2\right)} \oplus \frac{i\,\mathbb{Z}}{M\left(\sin\phi_2\right)}\right),$$

where we have attached, and shall further attach, the index 2 in order to denote division by 2. While further observing that

$$2\Lambda_2 = \frac{\Lambda\left(\cos\phi_2\right)}{\sqrt{\cos\phi_2}} = \frac{\Lambda\left(\sin\phi_2\right)}{\sqrt{-\sin\phi_2}} = \frac{\Lambda\left(i\tan\phi_2\right)}{\sqrt{i\cos\phi_2\sin\phi_2}}, \tag{3}$$

one ought to keep in mind that

$$\frac{i\pi}{2}\left(\frac{1}{M\left(\sin\phi_2\right)} - \frac{1}{M\left(-\sin\phi_2\right)}\right) \in 2\Lambda_2\backslash\Lambda,$$

and, in particular, that the value

$$\mathcal{R}\left(\frac{i\pi}{4M\left(\pm\sin\phi_2\right)},\ \beta = e^{i\phi}\right) = -1$$

does not depend upon the choice of the sign, appearing in the denominator of the first argument of the function $\mathcal{R}$.
The lattice relations (3) express the homothety invariance of lattices $\Lambda(\beta)$ as the square β^2 of their parameter undergoes the inversions (that is linear fractional transformations of order 2):

$$R : x \mapsto \frac{1}{x},\ L : x \mapsto 1 - x.$$

These two inversions generate a (6 element) group isomorphic with the symmetry group S_3 of a triangle. The three pairs

$$\{-\tan^2, -\cot^2\},\ \{\sin^2, \cos^2\},\ \{\csc^2, \sec^2\}$$

might be viewed as the three vertices, which are rotated via either the composition $R \circ L$ or its inverse $L \circ R$. The first vertex is invariant under the action of R which transposes the second vertex with the third, while the second vertex is invariant under the action of L which transposes the third vertex with the first, and the third is invariant under the action of

$$R \circ L \circ R = L \circ R \circ L : x \mapsto \frac{x}{x-1}$$

which transposes the first vertex with the second. Relevant details and consequences are given in [5].

3 The motion of a simple pendulum

The configuration space of a simple pendulum, being a circle, shall be identified with the unit circle $\mathbb{T}$. The motion of the pendulum might then be viewed as a motion of a point particle (holonomically) being restricted to lie in the configuration space, and being subjected to a uniform force field (which we might refer to as gravitational). We might furthermore assume the time unit as chosen as to presume the

[2] Strictly speaking (since the square root function is not single-valued), we can only claim that the two-element sets $\{w_\pm\}$ and $\{u_+ \mp u_-\}$ coincide with one and the same (unordered) pair.

(constant) acceleration, induced by this gravitational field, being set to a unit value. The position of the pendulum at $1 = e^0$ is then the stable equilibrium position (possessing least potential energy), whereas the position at $-1 = e^{i\pi}$ is the unstable equilibrium position. The period of the pendulum would then be made to coincide with the real period of an elliptic function which would map the real axis $\mathbb{R}$ to the afore indicated unit circle $\mathbb{T}$. In the classical sense, the real axis $\mathbb{R}$ would represent the time,[3] whereas the unit circle $\mathbb{T}$ would represent the configuration space, so that the said elliptic function would then match a current time with the corresponding current position of the pendulum.

The equation of motion, of a simple pendulum, stems from the law of energy conservation which might be written as

$$\frac{\dot{\theta}^2}{2} - \cos\theta = -\cos\phi, \tag{4}$$

where the function $\theta = \theta(t)$ is viewed as a current argument (as a function of the time variable t), corresponding to a pendulum current position $e^{i\theta} =: \mathcal{E} = \mathcal{E}(t) = \mathcal{E}(t,\, \phi)$, with the angle θ being restricted to lie within a range, determined by *the angle of the maximal inclination* ϕ, that is $-\phi \leq \theta \leq \phi$.[4] Solving the energy conservation equation (4) entails expressing the (elliptic) function $\mathcal{E}$ as a function obeying this equation for all values of its time argument t. Although, t is usually restricted to be real-valued, we shall emphasize here that no such restriction needs be ever imposed. Actively lifting this (artificial) restriction is, in fact, necessary as we proceed to delve into analytic continuation of solutions and to switching from oscillatory to rotary motion.

One might readily verify that the energy conservation law (4) is obeyed by the (dual) functions:

$$\mathcal{E}_\pm = \mathcal{E}_\pm(t) = \mathcal{E}_\pm(t,\, \phi) = -\mathcal{R}\left(\frac{t + u_\mp(\phi)}{2},\, e^{i\phi}\right) = \mathcal{R}\left(\frac{t + u_\mp(\phi)}{2\,i},\, -e^{i\phi}\right), \tag{5}$$

where the function $\mathcal{E}_+$ would turn out to agree with the function $\mathcal{E}$, which we had defined as the function mapping the current time t to the current pendulum position $e^{i\theta}$. The half-period $u_-(\phi)$ of $\mathcal{R}\left(\cdot, e^{i\phi}\right)$, appearing in two successive expressions on the right hand side, ensures the (natural) choice of the initial condition $\mathcal{E}(0,\, \phi) = 1$. That half-period is added (rather than subtracted) so as to ensure that the angle θ is moving towards ϕ (rather than towards $-\phi$) at the initial instant $t = 0$, and (since $\mathcal{R}$ is an even function) reversing the time corresponds to reversing the direction of motion, that is $\mathcal{E}(t,\, \phi) = \mathcal{E}(-t,\, -\phi)$. The (imaginary) half-period $u_-(\phi)$ can be calculated, along with the (real) half-period $u_+(\phi)$, via applying the second formula of the pair (2):

$$u_+(\phi) = u\left(e^{i\phi}\right) = \frac{\pi}{2M\left(\cos\phi_2\right)},\;\; u_-(\phi) = i\,u\left(-e^{i\phi}\right) = \frac{i\,\pi}{2M\left(\sin\phi_2\right)}.$$

As we arrive at discussing concrete examples, the significance of the (dual) function $\mathcal{E}_-$ will become apparent, but we might already indicate its relevance in extending the configuration space of the pendulum, as demonstrated in fig. 1.

The half-periods $u_\pm$ (of $\mathcal{R}$) constitute, of course, quarter-periods of $\mathcal{E}$. The time required for a (single) swing of the pendulum, that is the period, in the oscillatory case, is $4u_+$, whereas the time required for a (single) revolution of the pendulum, that is the period, in the rotary case, is $2w_+$.

The conservation of energy law (4) might be rewritten as

$$(\dot{\theta_2})^2 + (\sin\theta_2)^2 = (\sin\phi_2)^2,$$

in order to obtain a classical well-known solution of the latter equation, expressing the function

$$z := \csc\phi_2\,\sin\theta_2$$

as an elliptic function of the variable t. Explicitly,

$$z = z(t) = \mathrm{sn}\left(t,\, \sin\phi_2\right), \tag{6}$$

where $\mathrm{sn}\left(\cdot,\, k\right)$ is the Jacobi elliptic sine function with an elliptic modulus k.

The real quarter-period u_+ of the function z, with an elliptic modulus $\sin\phi_2$, might (consistently) be defined as

$$u_+ := \int_0^1 \frac{dz}{(1 - z^2)(1 - (\sin\phi_2)^2 z^2)},$$

[3]We might already observe here that the point $t = \infty$ would customarily be excluded from our function's domain. Such an omission is neither innocent nor insignificant, as we shall soon discover!

[4]This inequality is readily extendable to the rotary case, as soon as we find out that π (or $-\pi$) would then constitute the real part of ϕ.

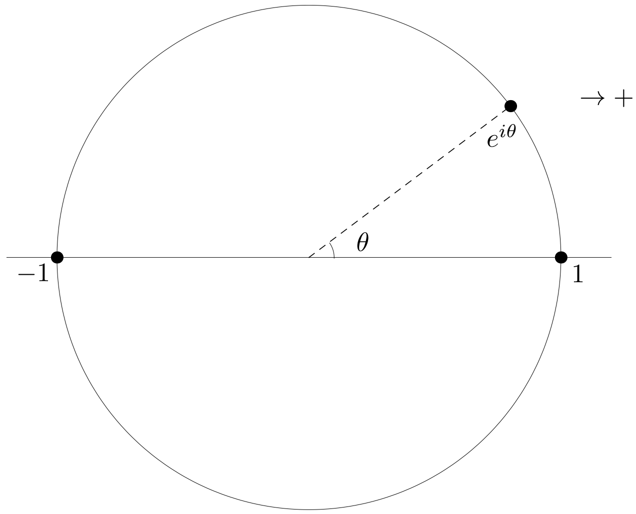


Figure 1: The union of the unit circle $\mathbb{T}$ with the real line $\mathbb{R}$ (necessarily compactified with the point at ∞) are exhibited as the "extended" configuration space of the pendulum. The closed ray $[-\infty,\, 0]$, in particular, is the range of the periodic function $\mathcal{E}_-(t,\, 0)$, for $t \in [0,\, 2\pi]$, giving rise, via the functional relation (8), to Abrarov's solution. Three positions are marked: the two points of the intersection $\mathbb{T} \cap \mathbb{R}$ are the equilibrium points, the stable equilibrium at 1 and the unstable equilibrium at -1, the third point at $e^{i\theta}$ has the angle θ as its argument (as shown). The pendulum is presumably "immersed" in an external (gravitational) force field of constant (unit) magnitude and direction, coinciding with the positive direction of the real axis.

so $u_+ = u_+(\phi)$, as before, whereas the complementary quarter period of the same function z, with the same elliptic modulus $\sin\phi_2$, might be denoted by u_-, where $u_- := i\, u_+(\pi - \phi) = i\, u\left(-e^{i\phi}\right)$.[5] Thus, and again, $u_- = u_-(\phi)$.

Both periods $u_\pm$ of the elliptic function z are shared with the elliptic function

$$v := \csc\phi_2\ \dot{\theta}_2,$$

which, analogously with expression (6), might be explicitly expressed as a function of t:

$$v = v(t) = \operatorname{cn}(t,\, \sin\phi_2) = \operatorname{sn}\left(\cos\phi_2\,(t - u_+),\, i\tan\phi_2\right).$$

In other words, the differential equation

$$\dot{y}^2 = \left(1 - y^2\right)\left(1 - k^2 y^2\right)$$

is satisfied by the elliptic functions z and $v \circ \sec$ (where the sign $\circ$ is used to denote the composition of the two functions surrounding it, with the function on its right hand side being first applied), corresponding to elliptic moduli $k = \sin\phi_2$ and $k = i\tan\phi_2$, respectively. So, in brief,

$$\dot{\theta}_2 = \sin\phi_2 \operatorname{cn}(t,\, \sin\phi_2)\,,\ \ \sin\theta_2 = \sin\phi_2 \operatorname{sn}(t,\, \sin\phi_2)\,. \tag{7}$$

Now, introduce an *alternative elliptic function*

$$\mathcal{S}(t,\, k) := \sqrt{k}\ \operatorname{sn}\left(\frac{t}{\sqrt{k}},\, k\right),^{6}$$

and verify, as in [5], that

$$\mathcal{R}(t,\, \beta) = \mathcal{S}(t,\, -\beta)^{-2},$$

so

$$e^{i\theta_2} = \mathcal{S}\left(\frac{it - iu_-(\phi)}{2},\, e^{i\phi}\right) = i\,\mathcal{S}\left(\frac{t - u_-(\phi)}{2},\, -e^{i\phi}\right).$$

[5] We are deviating from the conventional terminology, where a complementary quarter-period pair is rather the pair $u_+(\phi)$ and $u_+(\pi - \phi)$, as calculated via the corresponding formula for the real half-period of the function $\mathcal{R}$. Such a pair is real-valued whenever ϕ is, but such restriction upon ϕ has been actively avoided, in order to enable (a single) pendulum motion formula, valid in both oscillatory and rotary modes.

[6] Note that the functional homothety is correctly defined without specifying a branch of the (two-valued) square root function. It (implicitly) relies upon the oddness of the function $\operatorname{sn}(\cdot,\, k)$ viewed, for fixed k, as a function of a single (first) argument.

Few special cases

The total energy of the pendulum is $2(\sin \phi_2)^2 = 1 - \cos\phi$, so we might classify the cases, for which we shall calculate exact periods, via their corresponding values of $\cos\phi$, which does not exceed 1 if the energy is presumed nonnegative. No troubles, however, arise if we permit negative energy values, which do emerge if we switch to imaginary time. This duality was discussed in [6, 4], but it might be succinctly expressed via the functional relation:

$$\mathcal{E}_+(t,\, \phi) = -\mathcal{E}_-(-i\, t,\, \pi - \phi), \tag{8}$$

which is evident from (5). Note that the dual solutions $\mathcal{E}_\pm$ satisfy two distinct (dual) initial conditions:

$$\mathcal{E}_\pm(0) = \pm 1.$$

We shall exploit the duality, as expressed via (8), in pairing the cases we next consider.

- The self-dual case $\cos\phi = 0$: The period parallelogram of the corresponding function $\mathcal{E}(\cdot,\, \pi/2)$, along with the function $\mathcal{R}(\cdot,\, i)$, is then a square, represented on the left hand side of fig. 2. The time required for a (full) swing of the pendulum is a double period of the function $\mathcal{R}$, that is

$$4\, u_+\left(\frac{\pi}{2}\right) = 2\,\pi/M\left(\cos\left(\frac{\pi}{4}\right)\right) = 2\sqrt{2}\, G\, \pi,$$

  where $G := 1/M\left(\sqrt{2}\right)$ is the (so called) Gauss constant.

- The dual (equilibria) cases $\cos\phi = \pm 1$: The assumption that ϕ is not a multiple of π is lifted here. While $\mathcal{E}(t,\, 0) = \mathcal{E}_+(t,\, 0) \equiv 1$ is indeed a solution, to the differential equation (4) with $\phi = 0$, which is known as *the stable equilibrium solution*, another solution is

$$\mathcal{E}_-(t,\, 0) = -\cot\left(\frac{2\, t - \pi}{4}\right)^2 = \frac{\sin t + 1}{\sin t - 1} = -(\sec t + \tan t)^2.$$

  The latter (seemingly irrelevant, due to the initial condition $\mathcal{E}_-(0,\, 0) = -1$) solution is routinely missed along with an undeniably relevant solution to the differential equation (4) with $\phi = \pi$:

$$\mathcal{E}(t,\, \pi) = \mathcal{E}_+(t,\, \pi) = -\mathcal{E}_-(-i\, t,\, 0) =$$
$$= -\coth\left(\frac{2\, t + i\, \pi}{4}\right)^2 = \frac{i - \sinh t}{i + \sinh t} = (\operatorname{sech} t + i \tanh t)^2.$$

  The elementary critical solution $\mathcal{E}(\cdot,\, \pi)$ might rightfully be called *Abrarov's solution.*[7] It has unjustly been ignored before it was brought to attention by Abrarov, and yet remains missing from all mechanics textbooks ever written! Abrarov's solution satisfies the initial condition $\mathcal{E}(0,\, \pi) = 1$, with the motion occurring along $\mathbb{T}_+$ for $t > 0$, reaching the unstable equilibrium position at $t = \infty$.[8] The (other) solution $\mathcal{E}_-(t,\, \pi) \equiv -1$, to the differential equation (4) with $\phi = \pi$, is known as *the unstable equilibrium solution.* Abrarov's solution $\mathcal{E}(\cdot,\, \pi)$, along with the solution $\mathcal{E}_-(\cdot,\, 0)$, is (singly) periodic. The (real) period 2π, of the function $\mathcal{E}_-(\cdot,\, 0)$, is (unsurprisingly) the period of the "small angle" pendulum.[9]

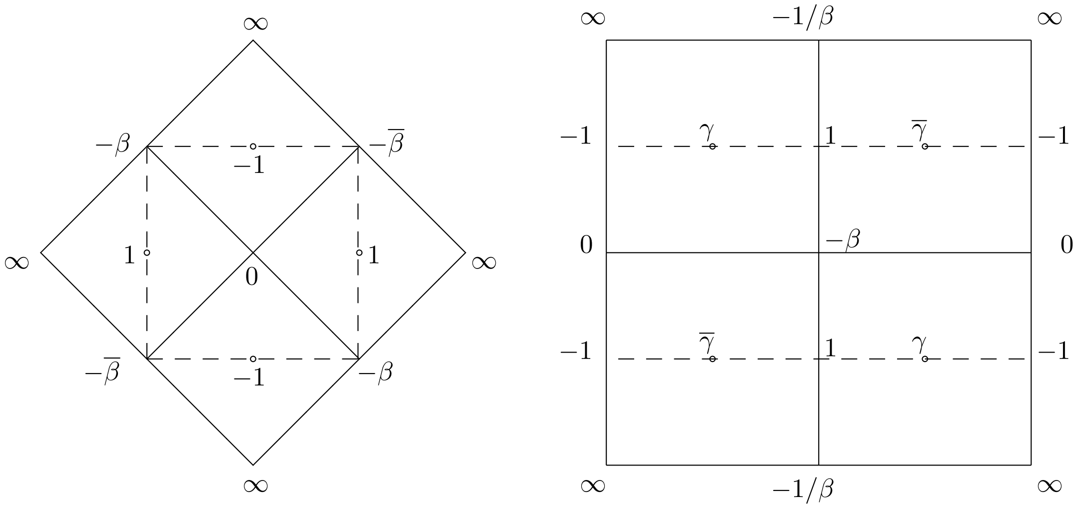
- The dual cases $\cos\phi = \pm 3/\sqrt{8}$: No loss of generality ensues if we restrict our attention to the case $\cos\phi = -3/\sqrt{8}$, where the total energy is nonnegative. The period parallelogram of the corresponding function $\mathcal{E}(\cdot,\, \pi + i/2 \ln 2)$, along with the function $\mathcal{R}(\cdot,\, -1/\sqrt{2})$, is (again) a square, represented on the right hand side of fig. 2. The time required for a single revolution of the pendulum is a (real) double period of $\mathcal{R}$, that is

$$2\, w\left(\frac{-1}{\sqrt{2}}\right) = 2\sqrt{\sqrt{2}}\, G\, \pi.$$

[7] Dmitry L. Abrarov (CCRAS, Moscow) has brought to my attention that the unstable equilibrium position is (wrongfully) excluded from the domain of definition of the conventional formulas of motion of the pendulum. Thus, oscillatory and rotary modes are customarily (artificially) dichotomized, leading to common misconceptions (upon which we shall not elaborate here) in interpreting the behavior of the motion of the pendulum in vicinity of the unstable equilibrium position. Abrarov's solution $\mathcal{E}(t,\, \pi) = \mathcal{E}_+(t,\, \pi)$ separates oscillatory solutions, with rhombic lattices, from rotary solutions, with rectangular lattices. It must be singled out as a first (and foremost) rotary case, much in analogy with singling out the stable equilibrium $\mathcal{E}(t,\, 0) \equiv 1$ whenever oscillatory motion is considered.

[8] Note that the point $t = \infty$, regarded as the (single) point compactification of $\mathbb{C}$, is the other point (along with $t = 0$) where imaginary and real values intersect (on the Riemann sphere).

[9] We remind the reader that the time unit was as chosen as to presume that its square coincides the ratio of the length of the pendulum to the magnitude of the acceleration. The conventional derivation of the period, of "small angle" pendulum, presumes that the values $\sin\theta$ and θ are "close" to each other.



$\beta = i$ $\qquad\qquad$ $\beta = -1/\sqrt{2}$

Figure 2: The rectangle and the rhombus (with the square being a special case for both) as period parallelograms of the essential elliptic function $\mathcal{R} = \mathcal{R}(\cdot, \beta)$, which parameter β lies on the boundary of $\mathbb{D}_+$. Along with the (double) poles at the vertices, the values of $\mathcal{R}$ at half-periods are indicated, as well as, the values ± 1 at their corresponding quarter-periods. Four more (quarter-period) values are indicated in the second (rectangular) case, where $\gamma = i\sqrt{1-\beta^2} - \beta \in \mathbb{T}_+$. The dashed lines are (two-fold) mapped, via $\mathcal{R}$, onto the unit circle $\mathbb{T}$, that is, the configuration space of the pendulum. Further details on this mapping are (graphically) provided in [2].

- The dual cases $\cos\phi = \pm 3$: Once again we restrict our attention to the case $\cos\phi = -3$, where the total energy is nonnegative. The period parallelogram of the corresponding function $\mathcal{E}(\cdot,\, \pi + 2\,i\ln(\sqrt{2}+1))$, along with the function $\mathcal{R}(\cdot,\, 2\sqrt{2}-3)$, is now a rectangle which length is twice its width. The time required for a single revolution is

$$2\,w\left(2\sqrt{2}-3\right) = G\,\pi.$$

4 Conclusion

The classical solution to the motion of the simple pendulum, as given by (7), is hardly satisfactory since it neither yields the position of the pendulum as an explicit function of time, nor is it (usually) adapted to rotary cases, where the Jacobi elliptic modulus is not real-valued. The solution, denoted by $\mathcal{E} = \mathcal{E}_+$, as given by (5), does provide the position of the pendulum as an explicit time-dependent elliptic function, which lattice is rhombic, in the oscillatory case, and rectangular, in the rotary case. Equilibria are seen as solutions separating oscillatory and rotary solutions, where the stable equilibrium solution $\mathcal{E}(t,\, 0) \equiv 1$ is a (foremost) oscillatory case, whereas Abrarov's solution $\mathcal{E}(t,\, \pi) = \mathcal{E}_+(t,\, \pi)$ is a (foremost) rotary case. The equilibria are, as well, the solutions "gluing" together oscillatory and rotary modes of motion in an analytic unifying formula (5), refuting the established belief that "a single analytic expression is not possible for describing the trajectories of motion of the pendulum in both oscillatory and rotary zones", as stated in [9, p. 256].[10] Without Abrarov's insight, providing the last necessary step towards the unifying solution, this basic problem of classical mechanics was not entirely solved, so in accordance with the principle "Nil actum reputans si quid superesset agendum" from [8, p. 629], it was not at all ever solved! May its solution be now declared on this wonderful occasion (on February 9, 2013) in celebration of our dearest Jan Jerzy Sławianowski 70^{th} birtday!

[10] That belief was "supported" by topological arguments, convincing those "specialists" who apparently confused the absence of an analytic unifying formula of motion, in both oscillatory and rotary regimens, of a simple pendulum by the end of the second millennium with its nonexistence.

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