The Galois top and its motion invariant

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Abstract

The notion of the Galois top, along with its motion invariant, was introduced on April 17, 2024 at the Polynomial Computer Algebra Annual Conference. Here we prove that the Galois top is well-defined. The Lagrange top and the Euler top are then seen as two special cases of the Galois top. We demonstrate that their "additional" invariants of motion (which is the constant projection of the angular momentum on the axis of symmetry, for the Lagrange top, and the constant modulus of the angular momentum, for the Euler top) are special cases of the Galois top "additional" invariant of motion.

Introductory concepts and definitions

A "rotation" is an isometry of a Euclidean space, possessing a fixed point. A rotation is assumed to be "proper", with the adjective "proper" occasionally added to emphasize that the said isometry is orientation-preserving, whereas an "improper" rotation would be orientation-reversing. Whether proper or improper, a rotation in $\mathbb{R}^n$ is representable by an orthonormal n by n matrix which determinant is either positive (for a proper rotation) or negative (for an improper rotation). Proper rotations of $\mathbb{R}^n$ which share a common fixed point form a Lie group, called the special orthogonal group of dimension n and denoted by $\mathrm{SO}(n)$, which binary operation is the (non-commutative) matrix multiplication.

We shall refer to a rigid body, possessing a fixed point, throughout its motion as a "top". The motion of a top might evidently be represented by a time-dependent curve in its configuration space which might evidently be identified with the Lie group $\mathrm{SO}(3)$. Such motion curve in $\mathrm{SO}(3)$ lifts to a section in the phase space of the top which is $\mathrm{T}^*\mathrm{SO}(3)$, that is, the cotangent bundle of $\mathrm{SO}(3)$, since the (instantaneous) angular velocity of the top lies in the cotangent space at the point (in its configuration space $\mathrm{SO}(3)$), corresponding to a given time, on its motion curve.

We shall apply the adjective "heavy" to a top in order to indicate that its motion is governed by a gravitational force field.

Consider the (rotational) motion of a top and denote its fixed point by O. Denote by $\boldsymbol{\omega}$ and $\boldsymbol{m}$ the pseudovectors of its angular velocity and angular momentum, respectively.

The pseudovectors $\boldsymbol{\omega}$ and $\boldsymbol{m}$ might be expressed in a (rotating) coordinate system which directing unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are aligned along the principal axes of inertia, passing through the point O,[1] that is,

$$\boldsymbol{\omega} = p\,\boldsymbol{i} + q\,\boldsymbol{j} + r\,\boldsymbol{k}\,,\ \ \boldsymbol{m} = A\,p\,\boldsymbol{i} + B\,q\,\boldsymbol{j} + C\,r\,\boldsymbol{k}, \tag{1}$$

[1] The principal axes of inertia need not pass through the centre of mass, contrary to a false, yet common belief notably shared, among others, by Richard Feynman.

where A, B and C are the principal momenta of inertia, corresponding to the principal directions $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, respectively.[2]

The expression for the angular momentum, given in (1), is based on the identities

$$\boldsymbol{m} = \int \boldsymbol{r} \times \boldsymbol{\omega} \times \boldsymbol{r}\, d\rho =^{3} \int r^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{r})\, \boldsymbol{r}\, d\rho = J\, \boldsymbol{\omega},^{4}\ J = (J^n_m),\ J^n_m := \int \delta^n_m r^2 - r_m r_n\, d\rho,^{5}$$

where $\boldsymbol{r}$ is the radius vector of an "infinitesimal" mass element $d\rho$, the (raw and column) indices m and n run over the values $1, 2, 3$ and δ^n_m is the Kronecker symbol, that is, δ^n_m vanishes, unless it acquires the unit value when $m = n$ (on the diagonal of J). Thus, the matrix J, expressing the tensor of inertia of a rigid body, with respect to an orthogonal basis, fixed in it, being real and symmetric,[6] is unitarily diagonalizable.

We point out that once the matrix I of the tensor of inertia of a rigid body, about its centre of mass G, has been calculated, the Huygens-Steiner theorem enables a swift calculation of the matrix J of the tensor of inertia about any other point O, as

$$J = I + \begin{pmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_3 r_1 \\ -r_1 r_2 & r_3^2 + r_1^2 & -r_2 r_3 \\ -r_3 r_1 & -r_2 r_3 & r_1^2 + r_2^2 \end{pmatrix} \int d\rho, \quad \boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} := O - G.^{7}$$

Note that the principal axes of inertia about the point O do not (in general) coincide with the principal axes of inertia about the centre of mass G. However, if O lies on a principal axis through G then the directions of the principal axes through O would coincide with the directions of the principal axes through G.[8]

Before we proceed with explicit calculations, we need to note a relevant construction of an ellipsoid of inertia, as conceived by James MacCullagh [9]. The MacCullagh ellipsoid of inertia has the lengths of its (three) principal axes directly proportional to the corresponding square roots of the (three) principal momenta. Naturally, we might associate a MacCullagh ellipsoid of inertia with a given top, assuming the centre of the ellipsoid to coincide with the fixed point O of the top. The principal axes of the ellipsoid are then assumed to be aligned along the principal axes of inertia through O. We must also single out the top which fixed point coincides with its centre of mass G. For this top, an axis, orthogonal to the circular sections of MacCullagh ellipsoid has been called a "Galois axis" [3, 1, 2, 4, 5, 13, 14, 16],[9] and we shall refer to a chosen Galois axis as "the" Galois axis.

Calculating the tensor of inertia along the Galois axis

Suppose that the three principal momenta of inertia A, B and C, with respect to the centre of mass G, are pairwise distinct and lexicographically ordered, so that $A < B < C$.[10] The matrix,

[2] We note that a reversal in the ordering of momenta of inertia (ascendingly or descendingly) corresponds to swapping the chirality between a "right-handed" and a "left-handed" coordinate system.

[3] Although the cross product is non-associative we need not place brackets in this special (associative) case.

[4] We use the same letter (bolded and not) to (respectively) denote both a vector and its modulus.

[5] We have tacitly assumed an orthogonal coordinate system, where r_1, r_2 and r_3 are the coordinates of the vector $\boldsymbol{r}$.

[6] Note, moreover, that J is positive definite.

[7] The coordinates r_1, r_2 and r_3 of the radius vector $\boldsymbol{r}$ are presummed to be calculated in the same coordinate system, centred at G, in which I was caculated. Thus, the calculation of J merely requires the coordinates of O (in the said coordinate system) and body's (total) mass $\int d\rho$.

[8] That is, the (unordered) set of directions of the principal axes is preserved in this special case.

[9] A triaxial ellipsoid possesses two distinct "families" of circular sections.

[10] Recall that A, B and C are the eigenvalues of a positive definite matrix, so they are strictly positive.

corresponding to the tensor of inertia in a coordinate system which axes match the principal axes of inertia is diagonal. Let I denote this matrix, that is,

$$I = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix},$$

then according to the Huygens-Steiner theorem, the matrix of the tensor of inertia about the point

$$O = O(d) := \left(\sqrt{\frac{C(A-B)}{B(A-C)}},\ 0,\ \sqrt{\frac{A(B-C)}{B(A-C)}} \right) d, \tag{2}$$ [11]

which lies along the Galois axis [3, 2],[12] is the matrix

$$J = J(d) := \begin{pmatrix} A + \frac{A(B-C)\,d^2}{B(A-C)} & 0 & \frac{\sqrt{CA(A-B)(B-C)}\,d^2}{B(A-C)} \\ 0 & B + d^2 & 0 \\ \frac{\sqrt{CA(A-B)(B-C)}\,d^2}{B(A-C)} & 0 & C + \frac{C(A-B)\,d^2}{B(A-C)} \end{pmatrix},$$ [13]

which is diagonalizable by the unitary matrix

$$U := \begin{pmatrix} \alpha^+ & 0 & -\alpha^- \\ 0 & 1 & 0 \\ \alpha^- & 0 & \alpha^+ \end{pmatrix},$$

where

$$\alpha^+ := \frac{1}{2}\sqrt{\frac{(C-A+s)^2 - d^4}{(C-A)s}} = \sqrt{\frac{1+g/s}{2}},\ \alpha^- := \frac{1}{2}\sqrt{\frac{d^4-(C-A-s)^2}{(C-A)s}} = \sqrt{\frac{1-g/s}{2}},$$

$$s := \sqrt{(C-A)^2 + 2(C-A)h\,d^2 + d^4} = \sqrt{f^2 + \frac{4CA(A-B)(B-C)}{B^2}} =$$

$$= \sqrt{g^2 + \frac{4CA(A-B)(B-C)\,d^4}{B^2(C-A)^2}} = \sqrt{fg + \frac{2(A-CA/B)\left((C-A)^2 + 2(A-CA/B)d^2 + d^4\right)}{C-A}} =$$

$$= \sqrt{fg + (1-h)\left((C-A)^2 - (C-A)(1-h)\,d^2 + d^4\right)},$$

$$h := \frac{C+A-2CA/B}{C-A},\ g := C - A + h d^2,\ f := (C-A)h + d^2.$$

The columns of U are eigenvectors for J, so $\Lambda := U^{-1}JU$ is the diagonal matrix which (positive) entries are the corresponding eigenvalues of J:

$$\Lambda = \begin{pmatrix} \lambda_- & 0 & 0 \\ 0 & B+d^2 & 0 \\ 0 & 0 & \lambda_+ \end{pmatrix},\ \lambda_\mp := \frac{C + A + d^2 \mp s}{2}.$$ [14]

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[11]From now on, we assume to be taking the nonnegative branch of the (double-valued) square root of a nonnegative number.

[12]The directing unit vector along the "other" Galois axis is

$$\left(-\sqrt{\frac{C(A-B)}{B(A-C)}},\ 0,\ \sqrt{\frac{A(B-C)}{B(A-C)}} \right).$$

The (two) Galois axes are distinct as long as the momenta of inertia are pairwise distinct.

[13]Upon assuming a unit (total) mass, d^2 is then implicitly multiplied by this unit mass.

[14]The same eigenvalues are obtained if O was on the "other" Galois axis. The off-diagonal enries of the matrix J would then flip their signs (from negative to positive) and the eigenvectors would then become the columns of U^{-1} (which is the transpose of U). That would correspond to flipping the sign of α^- (exclusively) or to "flipping the chirality" of the coordinate system (which might, in particular, be done by reversing the (initially chosen) order of the momenta of inertia, as was pointed out in footnote 2).

We now calculate the "new" coordinates of the directing unit vector along "the Galois axis":[15]

$$\left(\sqrt{\frac{\lambda_+(B+d^2-\lambda_-)}{(B+d^2)(\lambda_+-\lambda_-)}},\,0,\,\sqrt{\frac{\lambda_-(\lambda_+-B-d^2)}{(B+d^2)(\lambda_+-\lambda_-)}}\,\right)=\left(\sqrt{\frac{1+f/s}{2}},\,0,\,\sqrt{\frac{1-f/s}{2}}\,\right),$$

and since

$$h=\frac{C+A-2CA/B}{C-A}=1-\frac{2A(C-B)}{B(C-A)}=$$

$$=1-\frac{2A(C-B)\left(B(C-A)^2-2A(C-B)d^2+Bd^4\right)+4CA(A-B)(B-C)d^2}{B^2(C-A)s^2}=$$

$$=\frac{fg-\sqrt{(s^2-f^2)\,(s^2-g^2)}}{s^2},$$

the "old" coordinates of the latter (unit) vector are thus

$$\frac{1}{2}\left(\sqrt{\left(1+\frac{f}{s}\right)\left(1+\frac{g}{s}\right)}-\sqrt{\left(1-\frac{f}{s}\right)\left(1-\frac{g}{s}\right)},\,0,\,\sqrt{\left(1+\frac{f}{s}\right)\left(1-\frac{g}{s}\right)}+\sqrt{\left(1-\frac{f}{s}\right)\left(1+\frac{g}{s}\right)}\,\right)=$$

$$=\left(\sqrt{\frac{1+fg/s^2-\sqrt{(1-f^2/s^2)(1-g^2/s^2)}}{2}},\,0,\,\sqrt{\frac{1-fg/s^2+\sqrt{(1-f^2/s^2)(1-g^2/s^2)}}{2}}\,\right)=$$

$$=\left(\sqrt{\frac{1+h}{2}},\,0,\,\sqrt{\frac{1-h}{2}}\,\right)=\left(\sqrt{\frac{C(A-B)}{B(A-C)}},\,0,\,\sqrt{\frac{A(B-C)}{B(A-C)}}\,\right).$$

We have thereby proven that the definition of the Galois axis need not be altered if the top, which fixed point is G, is replaced with a top which fixed point O lies anywhere on the chosen Galois axis, although the principal axes, corresponding to extreme momenta of inertia, of MacCullagh ellipsoid do rotate about the principal axis, corresponding to the intermediate moment of inertia.[16] We now, moreover, know that as we move further away from the centre of mass G, along the Galois axis, the principal axis, corresponding to the minimal moment of inertia, rotates towards the Galois axis but never reaches it. The ratio of the maximal to the intermediate moment of inertia tends (descendingly) towards unity as the distance from G to O (unboundedly) increases, whereas (concomitantly) the minimal moment of inertia (ascendingly) approaches the (fixed) value of the moment of inertia about the Galois axis:

$$\frac{\lambda_+}{B+d^2}\searrow 1,\ \lambda_-\nearrow\frac{CA}{B}=\frac{\lambda_+\lambda_-}{B+d^2}.$$

Special cases[17]

The Galois axis, in the case of the Lagrange top, that is, the case of dynamically symmetric rigid body, would coincide with the principal axis, corresponding to the (only) extreme (whether minimal or maximal) moment of inertia,[18] so we either have (at the centre of mass) $A=B\leq C$ or $A<B=C$.

[15]We mean the coordinates in which the tensor of inertia about the point O is diagonal. We are yet to discover that "the newly emerging" axis, through O, which is orthogonal to the circular sections of MacCullagh ellipsoid will pass through G, confirming it to be the Galois axis.

[16]Recall that the length of a principal axis of MacCullagh ellipsoid is proportional to the square root of the corresponding moment of inertia. We note that the direction of the principal axis, corresponding to the intermediate moment of inertia, does not depend on the location of O along the Galois axis.

[17]Including those, where the requirement upon the momenta of inertia (at the centre of mass) to be pairwise distinct is no longer imposed.

[18]The two Galois axes merge here with that principle axis.

- $A = B < C$, so $h = -1$ and

$$s = \begin{cases} g = -f = C - B - d^2 & \text{if } d^2 \le C - B \\ f = -g = B - C + d^2 & \text{if } d^2 \ge C - B \end{cases}$$

As O moves further away from G, the value of s diminishes then turns to zero (at $d^2 = C - B$)[19] before increasing. In other words, the matrix

$$J = \begin{pmatrix} B + d^2 & 0 & 0 \\ 0 & B + d^2 & 0 \\ 0 & 0 & C \end{pmatrix}$$

is scalar at $d^2 = C - B$, whereas the moment of inertia (about the Galois axis) C is maximal for $d^2 < C - B$ and minimal for $d^2 > C - B$.

- $A < B = C$, so $h = 1$ and $s = g = f = B - A + d^2$. As O moves further away from G, the value of s (strictly) increases. In other words, the matrix

$$J = \begin{pmatrix} A & 0 & 0 \\ 0 & B + d^2 & 0 \\ 0 & 0 & B + d^2 \end{pmatrix}$$

is never scalar and the moment of inertia (about the Galois axis) A is always minimal.

We shall also single out the case when B is the harmonic mean of C and A.

- $C > A$ and $B = \frac{2CA}{C+A}$, so $h = 0$ and $s = \sqrt{(C-A)^2 + d^4} = \sqrt{(C+A)(C+A-2B) + d^4}$. We observe that the minimal eigenvalue λ_- of the matrix

$$J = \begin{pmatrix} A + d^2/2 & 0 & -d^2/2 \\ 0 & B + d^2 & 0 \\ -d^2/2 & 0 & C + d^2/2 \end{pmatrix}$$

tends (ascendingly) to the arithmetic mean of C and A, which would then coincide with the moment of inertia about the Galois axis,[20] and since the Galois axes are orthogonal to each other in this case,[21] it must coincide with the greatest lower bound for the set of differences $\lambda_+ - d^2$, where λ_+ is the maximal eigenvalue of the matrix $J = J(d)$:

$$\lambda_- \nearrow \frac{C+A}{2}, \quad \lambda_+ \searrow \frac{C+A}{2} + d^2.$$

The definition of the Galois top

So, although the MacCullagh ellipsoid of inertia is transformed if we relocate its centre along the Galois axis, the orthogonality of the Galois axis to the circular sections is preserved. The two principal axes, corresponding to extreme momenta of inertia would rotate, whereas the direction of the principal axis, corresponding to the intermediate moment of inertia is preserved (remaining orthogonal to the Galois axis). And since the latter (newly emerging, rotated and rescaled) MacCullagh ellipsoid of inertia shares the "same" Galois axis with the former ellipsoid we might proceed to define the "Galois top" as follows.

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[19] Recall that we have assumed the top to possess a unit mass so that d^2 acquires the units of the moment of inertia upon multiplying it by the unit mass.

[20] We might recall here that the product of the arithmetic and harmonic means is the square of the geometric mean.

[21] We must emphasize that the said orthogonality of the Galois axes pertains (strictly) to the initial value of d, that is, for $d = 0$. In fact, the Galois axis must, by definition, pass through the centre of mass G, so one and only one Galois axis passes through the fixed point O whenever $d > 0$.

The "**Galois top**" is a heavy top, in a uniform gravitational field, which fixed point lies at the Galois axis.

We must emphasize that the Galois top possesses a single Galois axis whenever its fixed point O differs from its centre of mass G. The Galois axis (for the said Galois top) is then (uniquely) determined via the said points O and G. The Euler top is then seen as a Galois top which fixed point O coincides with its centre of mass G so that the Galois axis (in this exceptional case) is not necessarily uniquely determined.

We must also emphasize that the Galois axis does necessarily (by definition) pass through the centre of mass unlike any principal axis of inertia which is well-defined at any point and need not necessarily pass through the centre of mass. As already said, the Euler top is then the only exception for which the Galois axis (for the Galois top) is not necessarily unique.

The Galois top (thus defined), which fixed point is O, is dualizable since the Galois axis is. The fixed point $\bar{O}$ of the dual top would lie on the dual Galois axis at the same distance from centre of mass G as was the distance from O to G.

The Galois top invariant of motion

The coordinates of the centre of mass G of the Galois top with respect to a coordinate system which (directed) axes coincide with the principal axes of inertia, passing through the fixed point O, are

$$\left(\sqrt{\frac{C(A-B)}{B(A-C)}},\ 0,\ \sqrt{\frac{A(B-C)}{B(A-C)}}\right) d,$$

where d is the distance between the fixed point O and the centre of mass G, whereas A, B and C are (now) the (ascendingly ordered) principal momenta of inertia about the fixed point O (not G).[22]

Along with the well-known "classical" invariants of motion,[23] which are the (total) energy and the "vertical" projection of the angular momentum, the Galois top possesses its "own" invariant of motion, which is

$$e^{\gamma\alpha\int_0^t q(t)\,dt}\big(\gamma\, r(t)+\alpha\, p(t)\big)\equiv\gamma\, r(0)+\alpha\, p(0),\ \ \gamma:=\sqrt{\frac{C-B}{A}},\ \ \alpha:=\sqrt{\frac{B-A}{C}},^{24} \tag{3}$$

where p, q and r are the respective projections of the angular velocity on the (three) principal axes of inertia,[25] corresponding to the (principal) momenta $A\le B\le C$.[26]

[22] Note that the coordinates of G are now expressed via the principal momenta at O, unlike the coordinates of O, given in (2), which were expressed via the principal momenta at G. Yet, the fact that the two expressions (for the coordinates of O and G) do formally coincide, one with the other, is (of course) based on the notion of the Galois top (which we have proven to be well-defined).

[23] We use the term "invariant" of motion to indicate a constant function (of time), although an archaic term "first integral" is still abundantly used.

[24] We are tacitly assuming that γ and α, being defined as square roots, share the same (positive) sign, corresponding to a particular choice of the Galois axis. Choosing oposite signs would correspond to choosing the "other" Galois axis. Thus, the product $\gamma\alpha$ would be nonnegative for the first choice of the Galois axis and nonpositive for the other.

[25] That is, p, q and r are the time-dependent coordinate-functions of the angular velocity.

[26] Note that we are not stipulating that A, B and C are pairwise distinct. However, we are (temporarily) excluding the case $A=B=C$. Alternatively, the Galois top motion invariant might be expressed as

$$e^{-\bar\gamma\bar\alpha\int_0^t q(t)\,dt}\big(\bar\gamma\, r(t)+\bar\alpha\, p(t)\big)\equiv\bar\gamma\, r(0)+\bar\alpha\, p(0),\ \ \bar\gamma:=\sqrt{\frac{B-C}{A}},\ \ \bar\alpha:=\sqrt{\frac{A-B}{C}},\ \ A\ge B\ge C.$$

The Lagrange top and The Euler top are the two cases, where the Galois top coincides with its dual.

The Lagrange top arises when either $A = B$ (so $\alpha = 0$) or $B = C$ (so $\gamma = 0$) and, respectively, either r or p is a constant function (of time).

The Euler top arises when the fixed point O coincides with the centre of mass G, where the Galois axes intersect each other and the afore-indicated motion invariant must be coupled with its dual invariant:[27]

$$e^{\pm\gamma\alpha\int_0^t q(t)\,dt} g^{\pm}(t) \equiv g^{\pm}(0), \;\; g^{\pm}(t) := \gamma r(t) \pm \alpha p(t).$$

The (invariant) product of these two coupled invariants is a homogeneous (in r and p) binomial of degree 2, that is,

$$g^{+}(t)g^{-}(t) \equiv g^{+}(0)g^{-}(0) = \big(\gamma r(0)\big)^2 - \big(\alpha p(0)\big)^2 = \frac{m^2 - Bh}{CA},$$

where $h := Ap^2 + Bq^2 + Cr^2$ is twice the (conserved) energy and m is the (consequently constant) modulus of the angular momentum, that is, $m^2 := A^2p^2 + B^2q^2 + C^2r^2$.[28]

For the Dzhanibekov top, in particular, the projection of the angular momentum on one (and only one) of the Galois axes (identically) vanishes, whereas the projection on the "other" Galois axis "never" vanishes![29] Formally speaking, we have $|\gamma r(t)| \equiv |\alpha p(t)| > 0$; the duality of the Dzhanibekov top stemming (again) from the duality of the Galois axis. An intuitive determination of whether g_+ or g_- identically vanishes is based on the determination of "the" Galois axis which rotates uniformly throughout the critical rigid body motion [3]. Its determination was proposed by E.A. Mityushov in a letter (dated February 27, 2018) which was cited in [2, page 6].[30]

In order to prove that the function on the left-hand side of identity (3) does not (in fact) depend on time, we merely need to verify that its time-derivative (identically) vanishes, that is, we need to verify that

$$\gamma\alpha q(t)(\gamma r(t) + \alpha p(t)) + \gamma\dot{r}(t) + \alpha\dot{p}(t) = \alpha(\gamma^2 q(t)r(t) + \dot{p}(t)) + \gamma(\alpha^2 p(t)q(t) + \dot{r}(t)) \equiv 0. \qquad (4)$$

The latter expression still holds if we restore the previous ordering of the principal momenta $A \leq B \leq C$. We would then have $\bar{\gamma} = i\gamma$, $\bar{\alpha} = i\alpha$, where $i := \sqrt{-1}$, so $\bar{\gamma}\bar{\alpha} = -\gamma\alpha$. Most significantly, we ought to note that neither expression for the Galois top motion invariant is symmetric upon swapping the two directing vectors $\boldsymbol{k}$ and $\boldsymbol{i}$ (along the two extreme principal axes) which would correspond to (simultaneously) swapping the two (extreme) principal momenta C and A, as well as, swapping the two functions $r(\cdot)$ and $p(\cdot)$. However, one expression might be obtained from the other if we augment the said swapping of C and A with flipping the sign, appearing in the exponent. The said sign thus represents the orientation of the coordinate system (imposed upon the top), as we alluded to in footnote(2). The orientation is, of course, reversed (again) with reversing the direction of the vector $\boldsymbol{j}$ (along the intermediate principal axis) and (respectively) flipping the sign of the function $q(\cdot)$.

[27] The dual invariant of the dual top might evidently be obtained by flipping the sign of (exclusively) either γ or α. Less evidently, by flipping the sign of q along with flipping the sign of (exclusively) either r or p.

[28] Quite recently, that is, after submitting this article to the AiM journal, Nikita P. Kopytov reported on his implementation of the Galois top additional invariant in his dynamical testing of an IMU (inertial measurement unit), attached to a heavy, freely falling, rigid body [10].

[29] Assuming the complex plane to be the time domain. The latter assertion does not hold if the complex plane is compactified (to the Riemann sphere) via augmenting the time domain with the (single) point of complex infinity, as discussed in [13].

[30] A translation to English of that quote would be:

"In order to obtain the Galois axis, the minor axis must be rotated clockwise by the angle arccos $\sqrt{\frac{C(B-A)}{B(C-A)}}$, looking from the end of the angular momentum vector."

The latter identity (4) might be interpreted as a sustained (mutual) orthogonality of the exerted torque (due to gravity) and the Galois axis throughout the motion of the Galois top. The said orthogonality is, of course, entrenched in the Euler equations for the Galois top which we shall (next) derive.

Denote with $\boldsymbol{n}$ the unit vertical upwards (in the direction opposing the direction of gravity) and express it, via the Euler angles [6], as

$$\boldsymbol{n} = \sin\phi\sin\theta\,\boldsymbol{i} + \cos\phi\sin\theta\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k},$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors along the principal axes, corresponding to the principal momenta of inertia A, B and C (about the point O), respectively.

The exerted torque $\boldsymbol{\tau}$ (that is, the moment of external forces) is calculated as the cross product of the radius vector $\boldsymbol{d} = (g_1, 0, g_3)d$, eminating from the fixed point O towards the centre of mass G, and the weight of the top $\boldsymbol{w}$ (that is, the gravitational force):[31]

$$\boldsymbol{\tau} = w\,\boldsymbol{n} \times \boldsymbol{d},$$

so we might, accordingly,[32] write down the Euler equations as

$$\begin{aligned} A\,\dot{p} + (C - B)\,q\,r \quad &= g_3\cos\phi\sin\theta, \\ B\,\dot{q} + (A - C)\,r\,p \quad &= g_1\cos\theta - g_3\sin\phi\sin\theta, \\ C\,\dot{r} + (B - A)\,p\,q \quad &= -g_1\cos\phi\sin\theta. \end{aligned}$$

We now note that the unit vector $(g_1, 0, g_3)$ is colinear with the vector $(\alpha/A, 0, \gamma/C)$ (as both vectors lie along the Galois axis). Consequently, the "total" time-derivative of the angular momentum $d\boldsymbol{m}/dt =$[33] $\partial\boldsymbol{m}/\partial t + \boldsymbol{\omega} \times \boldsymbol{m}$ remains orthogonal to the (said) Galois axis throughout the motion of the Galois top. In other words, identity (4) holds, thereby proving identity (3).[34]

The two quantities which are twice the energy $\boldsymbol{\omega} \cdot \boldsymbol{m} + 2w\,\boldsymbol{d} \cdot \boldsymbol{n}$ and the "vertical" projection of the angular momentum $\boldsymbol{m} \cdot \boldsymbol{n}$ are, of course, also conserved for the Galois top, being a heavy top.

Conclusion

The mere existence of the Galois top as a natural generalization of both the Lagrange top and the Euler top refutes the fallacy of the hasty claim, made in [8, page 15],[35] concerning the problem of determining "additional" invariants of rigid body motion about a fixed point, allegedly "closed". The allegation was based on a paper [11] which abstract (in English) consists of the following two (false) statements:

[31] That gravitational force $\boldsymbol{w}$ is applied to the centre of mass. Its magnitude coincides with the magnitude of the weight, as defined by the third General Conference on Weights and Measures, in 1901 [15, page 46].

[32] Upon assuming that $O \neq G$ (so that $d \neq 0$) and that the units of distance and force are so now chosen so that the product $w\,d$ coincides with the unit of energy.

[33] The subsequent use of partial derivative notation is meant to signify the restriction of the differentiation to (only) the coordinates in a moving frame.

[34] For the Lagrange top, the coplanarity of $\boldsymbol{m}$, $\boldsymbol{\omega}$ and $\boldsymbol{d}$, along with identity (4), ensures the constancy of the dot product $\boldsymbol{m} \cdot \boldsymbol{d}$. This, of course, is a special case of the general case (of the Galois top) for which the time-derivative of $\boldsymbol{m} \cdot \boldsymbol{d}$ coincides with the determinant (that is, the scalar triple product) of (the ordered triple) $\boldsymbol{m}$, $\boldsymbol{\omega}$ and $\boldsymbol{d}$.

[35] That claim was made on page 32 in the Russian edition of 2001.

"If the ellipsoid of inertia is not an ellipsoid of revolution, the equations of motion are not integrable by Liouville quadratures. This result considerably strengthens the Poincaré-Husson theorem concerning the absence of an algebraic integral."[36]

The first statement does not, in any way, correspond to that cited paper [11]. It was apparently later made by its author (Kozlov V.V.) and it was (later yet) most beautifully refuted by the Dzhanibekov top which ellipsoid of inertia is not an ellipsoid of revolution, whereas its Galois axis rotates uniformly throughout its "flipping" motion, as discussed in [3, 1, 2, 4, 5, 13, 14]. The second statement (beginning with a demonstrative prnoun) does not follow from the first nor was the result "concerning the absence of an algebraic integral" strengthened. Spectacularly, the Galois top invariant of motion does not explicitly depend on Euler angles deeming "the second theorem", in [11], which allegedly "strengthens the Poincaré-Husson theorem" not only false but irrelevant. In fact, Poincaré never posed a problem concerning an absence of an additional (non-algebraic) invariant of motion. Moreover, Poincaré did caution his readers not to rush to conclusions,[37] as Kozlov V.V. unwisely did.[38]

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[2] Adlaj S. *Dzhanibekov screw.* Motivated by two talks delivered on October 26, 2016 and May 16, 2018 at the "V.A. Egorov Seminar on Space Flight Mechanics" at MSU, Moscow, Russia. Access link available at `https://semjonadlaj.com/Talks/SScrew.pdf`.

[3] Adlaj S. *Torque free motion of a rigid body: from Feynman wobbling plate to Dzhanibekov flipping wingnut.* Available at `https://semjonadlaj.com/SP/TFRBM.pdf`.

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[5] Adlaj S. *Modular equations and fundamental problems of classical mechanics.* January 31, 2019 Session of the A.A. Dorodnicyn Seminar "Methods for solving problems of Mathematical Physics". The presentation and its discussion are available at `https://www.youtube.com/playlist?list=PL8StSu9qOYd6vYHtDiS7JNrCbF8CgjfdF`. An English translation is available at `https://semjonadlaj.com/Talks/20190131T&E20210301.pdf`.

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[36]The term "integrable" in the first sentence means "solvable", whereas the term "integral" in the second sentence means "invariant".

[37]Poincaré was anticipating Sofja Kowalewskaya's (perhaps most significant) work to appear. Most unfortunately, for all of us, Kowalewskaya died before publishing it.

[38]"50 years later", Kozlov V.V. told the story of "the launch of his career" [12]. His page, at the site of the RAS, as archived on April 17, 2024, indicates that he "solved the Poincaré problem concerning the absence of new analytical integrals in the problem of rotation of a heavy asymmetric rigid body with a fixed point". All this is well summarized with the Russian idiom "поспешишь – людей насмешишь" which is often brutally (yet appropriately) translated as "haste makes waste".

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