Free rigid body rotation matrix

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Let $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ denote the unit vectors, directed along the principal axes of inertia of a rigid body, corresponding to the principal momenta A, B and C. The three coordinate functions p, q and r of the angular velocity $\boldsymbol{\omega} = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$ of such freely rotating rigid body are then determined for a given (conserved) angular momentum $\boldsymbol{m}$ and (twice the) energy $h := \boldsymbol{\omega} \cdot \boldsymbol{m} = Ap^2 + Bq^2 + Cr^2$.

Each row of the (orthonormal) matrix

$$S := \begin{pmatrix} \alpha_0 \cos\psi - \alpha_1 \sin\psi & \alpha_0 \sin\psi + \alpha_1 \cos\psi & Ap/m \\ \beta_0 \cos\psi - \beta_1 \sin\psi & \beta_0 \sin\psi + \beta_1 \cos\psi & Bq/m \\ \gamma_0 \cos\psi - \gamma_1 \sin\psi & \gamma_0 \sin\psi + \gamma_1 \cos\psi & Cr/m \end{pmatrix},$$

$$\begin{pmatrix}\alpha_0\\ \beta_0\\ \gamma_0\end{pmatrix} := \frac{1}{\sqrt{\omega^2 - h^2/m^2}} \begin{pmatrix}(1 - Ah/m^2)\, p\\ (1 - Bh/m^2)\, q\\ (1 - Ch/m^2)\, r\end{pmatrix}, \quad \begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix} := \frac{1}{\sqrt{m^2\omega^2 - h^2}} \begin{pmatrix}(B - C)\, qr\\ (C - A)\, rp\\ (A - B)\, pq\end{pmatrix},$$

$$\psi = \psi(t) := \frac{ht}{m} + \left(\frac{h}{m} - \frac{m}{A}\right)\left(\frac{h}{m} - \frac{m}{B}\right)\left(\frac{h}{m} - \frac{m}{C}\right)\int_0^t \frac{dt}{\omega^2 - h^2/m^2},$$ [1]

which we calculated in [1], provides the three coordinates of a corresponding rotating unit vector with respect to an inertial (that is, a non-rotating) frame which directing unit vectors shall be denoted as $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z} := \boldsymbol{m}/m$.[2]

Accordingly, each column of the matrix S provides the three coordinates of a corresponding non-rotating unit vector with respect to the rotating frame. Thus, the matrix S might formally be construed via the relation

$$\begin{pmatrix}\boldsymbol{i}\\ \boldsymbol{j}\\ \boldsymbol{k}\end{pmatrix} = S \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\end{pmatrix}.$$

We might also readily express the angular velocity $\boldsymbol{\omega}$, with respect to the non-rotating frame, as $\boldsymbol{\omega} = \sqrt{\omega^2 - h^2/m^2}\,(\cos\psi\,\boldsymbol{x} + \sin\psi\,\boldsymbol{y}) + h/m\,\boldsymbol{z}$.

Acknowledgment

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References

[1] Adlaj S. *Torque free motion of a rigid body: from Feynman wobbling plate to Dzhanibekov flipping wingnut.* Available at `https://semjonadlaj.com/SP/TFRBM.pdf`.

[2] Kopytov N.P. *Using wireless sensors to monitor rotational motion of a "free flight" of a rigid body.* July 30, 2024 Session of the V.V. Shevchenko Seminar on Algebraic Methods in Theoretical Mechanics. Available at `https://www.mathnet.ru/php/seminars.phtml?presentid=43777&option_lang=eng`.

[1] We have used the non-boldface letters ω and m to denote the moduli of the angular velocity and the angular momentum, respectively, so $\omega^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega} = p^2 + q^2 + r^2$ and $m^2 = \boldsymbol{m} \cdot \boldsymbol{m} = A^2p^2 + B^2q^2 + C^2r^2$.

[2] We have tacitly excluded the case where $\boldsymbol{\omega}$ and $\boldsymbol{m}$ are colinear, that is, the case of the vanishing difference $\omega^2 - h^2/m^2$, including the critical motion of the Dzhanibekov top which was addressed in [1]. Note that the condition $(Ah - m^2)(Bh - m^2)(Ch - m^2) = 0$ implies that ψ is linear (as a function of time) although it does not imply the constancy of the angular speed ω, whereas the condition $(A - B)(B - C)(C - A) = 0$ does imply the constancy of ω along with the linearity of ψ.