

The suitcase of Königsberg

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This paper is inspired by a monumental construction of the regular 65537-gon by Hermes. His immense manual calculations, which took well over a decade, might nowadays be reproduced within a few days of merely organizing automated computations.

Introduction and statement

Johann Gustav Hermes (1846–1912) assembled his 221-large-format-page testament of the regular 65537-gon construction in a suitcase which is currently located at the University of Göttingen. The construction itself, contrary to a common belief refuted by Nikolai Vavilov [2], was carried out in Königsberg (now Kaliningrad), not Göttingen.¹ The construction was declared in 1894 under the title “Über die Teilung des Kreises in 65537 gleiche Teile” [1].

Let s_a^b denote the sum:

$$s_a^b := \sum_{k=1}^{32768/2^b} c(a + k2^b), \quad c(l) := 2 \cos\left(\frac{2\pi 3^l}{65537}\right),$$

where the number 3, appearing in the definition of the algebraic integer $c(\cdot)$, might be replaced with any other generator of the multiplicative group of integers modulo the prime 65537 without altering the sum, merely reordering its summands.

Note, in particular, that the sum s_0^b , which we shall for the sake of brevity denote with s^b , is the sum of the 2^b -th powers of a (fixed) primitive 65537th root of unity. So we have, $s^0 = -1$ and for each integer b , $0 \leq b \leq 14$, the sum s^{b+1} is a root of the quadratic polynomial

$$p_b(x) := x^2 - s^b x + s_{2^b}^{b+1} s^{b+1}.$$

Our task is accomplished once we express the algebraic integer $s^{15} = 2 \cos(2\pi/65537)$, which algebraic degree over $\mathbb{Q}$ is $2^{15} = 32768$, via nested quadratic radicals which we calculate via consecutively solving 15 quadratic equations.

We begin by observing that each product $s_{2^b}^{b+1} s^{b+1}$ “resides” in an extension field of $\mathbb{Q}$ which algebraic degree is 2^b . It, in fact, might be expressed as an integer linear combination of algebraic integers which algebraic degree, over $\mathbb{Q}$, does not exceed 2^b .

Expressing 15 products as integer linear combinations of algebraic integers

We successively calculate

$$s_1^1 s^1 = -16384,$$

$$s_2^2 s^2 = -4096,$$

$$s_4^3 s^3 = -1024 + 16(s_1^2 + s_3^2 - 2s^2) = -1024 + 16s_1^1 - 32s^2,$$

$$s_8^4 s^4 = -256 - 19s_1^3 + 16s_2^3 - 19s_3^3 + 13s_5^3 - 19s_7^3 + 28s^3 = -256 - 19s_1^1 + 16s_2^2 + 32s_5^3 + 28s^3,$$

¹Thus the title of our paper aims to emphasize the origin of the suitcase, not its current location.

$$\begin{aligned}
s_{16}^5 s^5 &= -64 - 2s_1^4 - 4s_2^4 - 7s_3^4 - 4s_4^4 - 3s_5^4 - 3s_6^4 - 4s_7^4 + 4s_8^4 - 6s_{11}^4 + s_{12}^4 + 6s_{13}^4 - 3s_{14}^4 + 6s_{15}^4 + 16s^{44}, \\
s_{32}^6 s^6 &= -16 - 4s_1^5 + 4s_2^5 - 3s_3^5 + 4s_4^5 + 2s_5^5 + 3s_7^5 + 3s_8^5 + 6s_9^5 - 4s_{10}^5 + 6s_{11}^5 - 3s_{12}^5 - 3s_{13}^5 - 5s_{14}^5 \\
&\quad + 6s_{15}^5 + 4s_{16}^5 - s_{17}^5 + 9s_{18}^5 - 4s_{19}^5 - 4s_{21}^5 + s_{23}^5 + 13s_{24}^5 - 9s_{26}^5 + s_{27}^5 - 3s_{28}^5 + s_{29}^5 - 3s_{30}^5 \\
&\quad - 5s_{31}^5 - 12s^{55}, \\
s_{64}^7 s^7 &= -4 - s_1^6 - 2s_2^6 + s_3^6 + s_4^6 + s_5^6 - 2s_6^6 + s_7^6 - 2s_8^6 + 2s_9^6 + s_{10}^6 + 2s_{11}^6 - 3s_{12}^6 + 2s_{13}^6 - s_{14}^6 \\
&\quad + s_{15}^6 + 2s_{16}^6 + 4s_{17}^6 - s_{18}^6 + s_{19}^6 - 2s_{20}^6 + s_{21}^6 + s_{22}^6 - 3s_{23}^6 - s_{24}^6 + 6s_{25}^6 - s_{26}^6 - s_{27}^6 + s_{29}^6 \\
&\quad - 3s_{30}^6 - 3s_{32}^6 + 2s_{33}^6 - s_{34}^6 - 2s_{35}^6 - 3s_{36}^6 - 3s_{37}^6 + 3s_{38}^6 - s_{39}^6 - s_{40}^6 - s_{41}^6 + 2s_{43}^6 - s_{44}^6 - 2s_{45}^6 \\
&\quad - s_{46}^6 + 3s_{48}^6 + 3s_{49}^6 + s_{52}^6 + 2s_{53}^6 + 4s_{54}^6 - 2s_{55}^6 - 2s_{56}^6 - s_{58}^6 - s_{60}^6 + s_{61}^6 + 4s_{62}^6 - s_{63}^6 - 4s^{66}, \\
s_{128}^8 s^8 &= s_1^7 + s_3^7 + s_4^7 + s_6^7 + s_7^7 + s_8^7 + 3s_{10}^7 + s_{14}^7 + s_{15}^7 + s_{16}^7 + s_{20}^7 + s_{22}^7 + s_{25}^7 + s_{26}^7 + 3s_{29}^7 + s_{30}^7 \\
&\quad + 2s_{33}^7 + 2s_{34}^7 + s_{35}^7 + 2s_{36}^7 + s_{37}^7 + 2s_{38}^7 + 2s_{40}^7 + 2s_{41}^7 + 2s_{42}^7 + s_{45}^7 + s_{47}^7 + s_{48}^7 + s_{51}^7 + s_{56}^7 \\
&\quad + 5s_{57}^7 + s_{58}^7 + 2s_{59}^7 + s_{60}^7 + s_{61}^7 + 2s_{63}^7 + 4s_{65}^7 + 2s_{67}^7 + 2s_{70}^7 + 2s_{73}^7 + 2s_{74}^7 + 2s_{76}^7 + 2s_{79}^7 + s_{80}^7 \\
&\quad + s_{82}^7 + 2s_{83}^7 + s_{86}^7 + 2s_{87}^7 + 2s_{88}^7 + s_{89}^7 + s_{90}^7 + 2s_{94}^7 + s_{95}^7 + 2s_{97}^7 + s_{98}^7 + s_{99}^7 + 3s_{100}^7 + 2s_{101}^7 \\
&\quad + 2s_{103}^7 + 2s_{104}^7 + 3s_{105}^7 + 4s_{106}^7 + 2s_{108}^7 + 2s_{109}^7 + s_{110}^7 + s_{111}^7 + 4s_{114}^7 + s_{115}^7 + s_{116}^7 + 2s_{117}^7 \\
&\quad + s_{118}^7 + s_{119}^7 + 2s_{121}^7 + s_{123}^7 + s_{124}^7 + s_{125}^7 + 2s_{127}^7 + 2s^{77}, \\
s_{256}^9 s^9 &= 2s_3^8 + s_4^8 + s_7^8 + s_{14}^8 + s_{19}^8 + 2s_{28}^8 + s_{29}^8 + s_{30}^8 + s_{33}^8 + s_{37}^8 + s_{39}^8 + s_{41}^8 + s_{43}^8 + s_{44}^8 + s_{50}^8 + s_{51}^8 \\
&\quad + s_{53}^8 + s_{56}^8 + s_{59}^8 + s_{61}^8 + s_{65}^8 + s_{68}^8 + s_{70}^8 + s_{78}^8 + s_{79}^8 + s_{81}^8 + 3s_{82}^8 + s_{84}^8 + s_{88}^8 + s_{89}^8 + s_{106}^8 \\
&\quad + s_{109}^8 + s_{112}^8 + s_{117}^8 + s_{124}^8 + s_{128}^8 + s_{135}^8 + s_{142}^8 + s_{146}^8 + 2s_{156}^8 + s_{173}^8 + s_{175}^8 + 2s_{185}^8 + s_{186}^8 \\
&\quad + s_{187}^8 + s_{188}^8 + s_{195}^8 + s_{200}^8 + s_{212}^8 + s_{219}^8 + s_{231}^8 + s_{233}^8 + s_{243}^8 + 2s_{245}^8 + s_{250}^8 + s_{252}^8 + s_{253}^8, \\
s_{512}^{10} s^{10} &= s_{40}^9 + s_{48}^9 + 2s_{67}^9 + s_{80}^9 + 2s_{85}^9 + 2s_{87}^9 + s_{91}^9 + s_{105}^9 + s_{108}^9 + s_{113}^9 + 2s_{134}^9 + 2s_{174}^9 + s_{210}^9 + s_{225}^9 \\
&\quad + s_{232}^9 + s_{268}^9 + s_{274}^9 + s_{277}^9 + s_{280}^9 + s_{302}^9 + s_{348}^9 + s_{378}^9 + s_{389}^9 + s_{430}^9 + 2s_{450}^9 + s_{464}^9, \\
s_{1024}^{11} s^{11} &= s_{23}^{10} + 2s_{154}^{10} + s_{184}^{10} + s_{308}^{10} + s_{359}^{10} + s_{530}^{10} + s_{666}^{10} + s_{718}^{10} + s_{733}^{10} + 2s_{777}^{10} + s_{840}^{10} + s_{945}^{10} + 2s^{10}, \\
s_{2048}^{12} s^{12} &= s_1^{11} + s_2^{11} + s_{777}^{11} + s_{800}^{11} + s_{1099}^{11} + s_{1178}^{11} + s_{1263}^{11} + s^{11}, \\
s_{4096}^{13} s^{13} &= s_1^{12} + s_{1265}^{12} + s_{1899}^{12} + s_{4003}^{12}, \\
s_{8192}^{14} s^{14} &= s_{3164}^{13} + s_{8100}^{13}, \\
s_{16384}^{15} s^{15} &= s_{3072}^{14}.
\end{aligned}$$

Note that upon expressing a product as a linear combination of algebraic integers, which algebraic degree over $\mathbb{Q}$ is 2^b , then the integer coefficients of such linear combination sum to 2^{14-b} .

Recursive expressions for discriminants and roots

We now go on to introduce the functions $r_b(\cdot)$, $1 \leq b \leq 15$, which respective domains are the integers modulo 2^b . We shall express a variable $\sigma = \sigma_b \dots \sigma_1$ of each function $r_b(\cdot)$ in a binary form where each of its b digits σ_i , $1 \leq i \leq b$, stands for a binary digit (0 or 1) in the binary representation of the variable σ , and we shall compute the function $r_{b+1}(\cdot)$ recursively as

$$r_{b+1}(\sigma_{b+1}\sigma_b \dots \sigma_1) = (-1)^{\sigma_{b+1} + \tau_b(\sigma_b \dots \sigma_1)} \sqrt{\frac{\Delta_b(\sigma_b \dots \sigma_1)}{4}} + \frac{r_b(\sigma_b \dots \sigma_1)}{2},$$

where τ is a binary sign-function on its b bits and Δ_b is the discriminant-function of its variable σ , regarded as an integer modulo 2^b , as we have already indicated.

Assuming that $r_0 \equiv s^0 = -1$, we might proceed with the recursive calculation the functions $r_b(\cdot)$, $1 \leq b \leq 15$, upon successively intermittently calculating the 15 discriminant-functions Δ_b , $0 \leq b \leq 14$, as

$$\Delta_0/4 = (r_0/2)^2 + 16384 = 65537/4,$$

$$\Delta_1(\sigma_1)/4 = (r_1(\sigma_1)/2)^2 + 4096,$$

$$\begin{aligned}
\Delta_2(\sigma_2\sigma_1)/4 &= (r_2(\sigma_2\sigma_1)/2)^2 + 1024 - 16r_1(\sigma_1 + 1) + 32r_2(\sigma_2\sigma_1), \\
\Delta_3(\sigma_3\sigma_2\sigma_1)/4 &= (r_3(\sigma_3\sigma_2\sigma_1)/2)^2 + 256 + 19r_1(\sigma_1 + 1) - 16r_3(\sigma_3\sigma_2\sigma_1 + 2) - 32r_3(\sigma_3\sigma_2\sigma_1 + 5) \\
&\quad - 28r_3(\sigma_3\sigma_2\sigma_1), \\
\Delta_4(\sigma)/4 &= (r_4(\sigma)/2)^2 + 64 + 2r_4(\sigma + 1) + 4r_4(\sigma + 2) + 7r_4(\sigma + 4) + 4r_4(\sigma + 5) + 3r_4(\sigma + 6) \\
&\quad + 4r_4(\sigma + 7) - 4r_4(\sigma + 8) + 6r_4(\sigma + 11) - r_4(\sigma + 12) - 6r_4(\sigma + 13) + 3r_4(\sigma + 14) \\
&\quad - 6r_4(\sigma + 15) - 16r_4(\sigma), \\
\Delta_5(\sigma)/4 &= (r_5(\sigma)/2)^2 + 16 + 4r_5(\sigma + 1) - 4r_5(\sigma + 2) + 3r_5(\sigma + 3) - 4r_5(\sigma + 4) - 2r_5(\sigma + 5) \\
&\quad - 3r_5(\sigma + 7) - 3r_5(\sigma + 8) - 6r_5(\sigma + 9) + 4r_5(\sigma + 10) - 6r_5(\sigma + 11) + 3r_5(\sigma + 12) \\
&\quad + 3r_5(\sigma + 13) + 5r_5(\sigma + 14) - 6r_5(\sigma + 15) - 4r_5(\sigma + 16) + r_5(\sigma + 17) - 9r_5(\sigma + 18) \\
&\quad + 4r_5(\sigma + 19) + 4r_5(\sigma + 21) - r_5(\sigma + 23) - 13r_5(\sigma + 24) + 9r_5(\sigma + 26) - r_5(\sigma + 27) \\
&\quad + 3r_5(\sigma + 28) - r_5(\sigma + 29) + 3r_5(\sigma + 30) + 5r_5(\sigma + 31) + 12r_5(\sigma), \\
\Delta_6(\sigma)/4 &= (r_6(\sigma)/2)^2 + 4 + r_6(\sigma + 1) + 2r_6(\sigma + 2) - r_6(\sigma + 3) - r_6(\sigma + 4) - r_6(\sigma + 5) \\
&\quad + 2r_6(\sigma + 6) - r_6(\sigma + 7) + 2r_6(\sigma + 8) - 2r_6(\sigma + 9) - r_6(\sigma + 10) - 2r_6(\sigma + 11) \\
&\quad + 3r_6(\sigma + 12) - 2r_6(\sigma + 13) + r_6(\sigma + 14) - r_6(\sigma + 15) - 2r_6(\sigma + 16) - 4r_6(\sigma + 17) \\
&\quad + r_6(\sigma + 18) - r_6(\sigma + 19) + 2r_6(\sigma + 20) - r_6(\sigma + 21) - r_6(\sigma + 22) + 3r_6(\sigma + 23) \\
&\quad + r_6(\sigma + 24) - 6r_6(\sigma + 25) + r_6(\sigma + 26) + r_6(\sigma + 27) - r_6(\sigma + 29) + 3r_6(\sigma + 30) \\
&\quad + 3r_6(\sigma + 32) - 2r_6(\sigma + 33) + r_6(\sigma + 34) + 2r_6(\sigma + 35) + 3r_6(\sigma + 36) + 3r_6(\sigma + 37) \\
&\quad - 3r_6(\sigma + 38) + r_6(\sigma + 39) + r_6(\sigma + 40) + r_6(\sigma + 41) - 2r_6(\sigma + 43) + r_6(\sigma + 44) \\
&\quad + 2r_6(\sigma + 45) + r_6(\sigma + 46) - 3r_6(\sigma + 48) - 3r_6(\sigma + 49) - r_6(\sigma + 52) - 2r_6(\sigma + 53) \\
&\quad - 4r_6(\sigma + 54) + 2r_6(\sigma + 55) + 2r_6(\sigma + 56) + r_6(\sigma + 58) + r_6(\sigma + 60) - r_6(\sigma + 61) \\
&\quad - 4r_6(\sigma + 62) + r_6(\sigma + 63) + 4r_6(\sigma), \\
\Delta_7(\sigma)/4 &= (r_7(\sigma)/2)^2 - r_7(\sigma + 1) - r_7(\sigma + 3) - r_7(\sigma + 4) - r_7(\sigma + 6) - r_7(\sigma + 7) - r_7(\sigma + 8) \\
&\quad - 3r_7(\sigma + 10) - r_7(\sigma + 14) - r_7(\sigma + 15) - r_7(\sigma + 16) - r_7(\sigma + 20) - r_7(\sigma + 22) \\
&\quad - r_7(\sigma + 25) - r_7(\sigma + 26) - 3r_7(\sigma + 29) - r_7(\sigma + 30) - 2r_7(\sigma + 33) - 2r_7(\sigma + 34) \\
&\quad - r_7(\sigma + 35) - 2r_7(\sigma + 36) - r_7(\sigma + 37) - 2r_7(\sigma + 38) - 2r_7(\sigma + 40) - 2r_7(\sigma + 41) \\
&\quad - 2r_7(\sigma + 42) - r_7(\sigma + 45) - r_7(\sigma + 47) - r_7(\sigma + 48) - r_7(\sigma + 51) - r_7(\sigma + 56) \\
&\quad - 5r_7(\sigma + 57) - r_7(\sigma + 58) - 2r_7(\sigma + 59) - r_7(\sigma + 60) - r_7(\sigma + 61) - 2r_7(\sigma + 63) \\
&\quad - 4r_7(\sigma + 65) - 2r_7(\sigma + 67) - 2r_7(\sigma + 70) - 2r_7(\sigma + 73) - 2r_7(\sigma + 74) - 2r_7(\sigma + 76) \\
&\quad - r_7(\sigma + 80) - r_7(\sigma + 82) - 2r_7(\sigma + 83) - r_7(\sigma + 86) - 2r_7(\sigma + 87) - 2r_7(\sigma + 88) \\
&\quad - r_7(\sigma + 89) - r_7(\sigma + 90) - 2r_7(\sigma + 94) - r_7(\sigma + 95) - 2r_7(\sigma + 97) - r_7(\sigma + 98) \\
&\quad - r_7(\sigma + 99) - 3r_7(\sigma + 100) - 2r_7(\sigma + 101) - 2r_7(\sigma + 103) - 2r_7(\sigma + 104) - 3r_7(\sigma + 105) \\
&\quad - 4r_7(\sigma + 106) - 2r_7(\sigma + 108) - 2r_7(\sigma + 109) - r_7(\sigma + 110) - r_7(\sigma + 111) - 4r_7(\sigma + 114) \\
&\quad - r_7(\sigma + 115) - r_7(\sigma + 116) - 2r_7(\sigma + 117) - r_7(\sigma + 118) - r_7(\sigma + 119) - 2r_7(\sigma + 121) \\
&\quad - r_7(\sigma + 123) - r_7(\sigma + 124) - r_7(\sigma + 125) - 2r_7(\sigma + 127) - 2r_7(\sigma), \\
\Delta_8(\sigma)/4 &= (r_8(\sigma)/2)^2 - 2r_8(\sigma + 3) - r_8(\sigma + 4) - r_8(\sigma + 7) - r_8(\sigma + 14) - r_8(\sigma + 19) \\
&\quad - 2r_8(\sigma + 28) - r_8(\sigma + 29) - r_8(\sigma + 30) - r_8(\sigma + 33) - r_8(\sigma + 37) - r_8(\sigma + 39) \\
&\quad - r_8(\sigma + 41) - r_8(\sigma + 43) - r_8(\sigma + 44) - r_8(\sigma + 50) - r_8(\sigma + 51) - r_8(\sigma + 53) \\
&\quad - r_8(\sigma + 56) - r_8(\sigma + 59) - r_8(\sigma + 61) - r_8(\sigma + 65) - r_8(\sigma + 68) - r_8(\sigma + 70) \\
&\quad - r_8(\sigma + 78) - r_8(\sigma + 79) - r_8(\sigma + 81) - 3r_8(\sigma + 82) - r_8(\sigma + 84) - r_8(\sigma + 88) \\
&\quad - r_8(\sigma + 89) - r_8(\sigma + 106) - r_8(\sigma + 109) - r_8(\sigma + 112) - r_8(\sigma + 117) - r_8(\sigma + 124) \\
&\quad - r_8(\sigma + 128) - r_8(\sigma + 135) - r_8(\sigma + 142) - r_8(\sigma + 146) - 2r_8(\sigma + 156) - r_8(\sigma + 173) \\
&\quad - r_8(\sigma + 175) - 2r_8(\sigma + 185) - r_8(\sigma + 186) - r_8(\sigma + 187) - r_8(\sigma + 188) - r_8(\sigma + 195) \\
&\quad - r_8(\sigma + 200) - r_8(\sigma + 212) - r_8(\sigma + 219) - r_8(\sigma + 231) - r_8(\sigma + 233) - r_8(\sigma + 243) \\
&\quad - 2r_8(\sigma + 245) - r_8(\sigma + 250) - r_8(\sigma + 252) - r_8(\sigma + 253),
\end{aligned}$$

$$\begin{aligned}\Delta_9(\sigma)/4 &= (r_9(\sigma)/2)^2 - r_9(\sigma + 40) - r_9(\sigma + 48) - 2r_9(\sigma + 67) - r_9(\sigma + 80) - 2r_9(\sigma + 85) \\ &\quad - 2r_9(\sigma + 87) - r_9(\sigma + 91) - r_9(\sigma + 105) - r_9(\sigma + 108) - r_9(\sigma + 113) - 2r_9(\sigma + 134) \\ &\quad - 2r_9(\sigma + 174) - r_9(\sigma + 210) - r_9(\sigma + 225) - r_9(\sigma + 232) - r_9(\sigma + 268) - r_9(\sigma + 274) \\ &\quad - r_9(\sigma + 277) - r_9(\sigma + 280) - r_9(\sigma + 302) - r_9(\sigma + 348) - r_9(\sigma + 378) - r_9(\sigma + 389) \\ &\quad - r_9(\sigma + 430) - 2r_9(\sigma + 450) - r_9(\sigma + 464),\end{aligned}$$

$$\begin{aligned}\Delta_{10}(\sigma)/4 &= (r_{10}(\sigma)/2)^2 - r_{10}(\sigma + 23) - 2r_{10}(\sigma + 154) - r_{10}(\sigma + 184) - r_{10}(\sigma + 308) \\ &\quad - r_{10}(\sigma + 359) - r_{10}(\sigma + 530) - r_{10}(\sigma + 666) - r_{10}(\sigma + 718) - r_{10}(\sigma + 733) \\ &\quad - 2r_{10}(\sigma + 777) - r_{10}(\sigma + 840) - r_{10}(\sigma + 945) - 2r_{10}(\sigma),\end{aligned}$$

$$\begin{aligned}\Delta_{11}(\sigma)/4 &= (r_{11}(\sigma)/2)^2 - r_{11}(\sigma + 1) - r_{11}(\sigma + 2) - r_{11}(\sigma + 777) - r_{11}(\sigma + 800) \\ &\quad - r_{11}(\sigma + 1099) - r_{11}(\sigma + 1178) - r_{11}(\sigma + 1263) - r_{11}(\sigma),\end{aligned}$$

$$\Delta_{12}(\sigma)/4 = (r_{12}(\sigma)/2)^2 - r_{12}(\sigma + 1) - r_{12}(\sigma + 1265) - r_{12}(\sigma + 1899) - r_{12}(\sigma + 4003),$$

$$\Delta_{13}(\sigma)/4 = (r_{13}(\sigma)/2)^2 - r_{13}(\sigma + 3164) - r_{13}(\sigma + 8100),$$

$$\Delta_{14}(\sigma)/4 = (r_{14}(\sigma)/2)^2 - r_{14}(\sigma + 3072).$$

We have thereby expressed each discriminant-function $\Delta_b(\cdot)$ via algebraic integers “residing” in an extension field of $\mathbb{Q}$ which algebraic degree is 2^b .

Explicit successive calculations of roots of quadratic polynomials

We might now explicitly calculate

$$\begin{aligned}r_1(\sigma_1) &= (-1)^{\sigma_1} \sqrt{\frac{\Delta_0}{4}} + \frac{r_0}{2} = \frac{(-1)^{\sigma_1} \sqrt{65537} - 1}{2}, \\ r_2(\sigma_2\sigma_1) &= (-1)^{\sigma_2+1} \sqrt{\frac{\Delta_1(\sigma_1)}{4}} + \frac{r_1(\sigma_1)}{2} = \frac{(-1)^{\sigma_2+1} \sqrt{131074 - 2(-1)^{\sigma_1} \sqrt{65537}} + (-1)^{\sigma_1} \sqrt{65537} - 1}{4}, \\ r_3(\sigma_3\sigma_2\sigma_1) &= (-1)^{\sigma_3+\sigma_2+\sigma_1+1} \sqrt{\frac{\Delta_2(\sigma_2\sigma_1)}{4}} + \frac{r_2(\sigma_2\sigma_1)}{2} = \\ &\quad \frac{(-1)^{\sigma_3+\sigma_2+\sigma_1+1} \sqrt{(\sqrt{65537} + (-1)^{\sigma_1} 255)(\sqrt{65537} - (-1)^{\sigma_2+\sigma_1} \sqrt{(65537 - (-1)^{\sigma_1} \sqrt{65537})/2})}}{4} \\ &\quad + \frac{(-1)^{\sigma_2+1} \sqrt{131074 - 2(-1)^{\sigma_1} \sqrt{65537}} + (-1)^{\sigma_1} \sqrt{65537} - 1}{8},\end{aligned}$$

but in order to proceed we are yet to address the most computationally intense task in constructing the regular 65537-gon which is the task of determining the binary sign-functions τ_b . So far we have $\tau_0 = 0$, $\tau_1(\sigma_1) \equiv 1$, $\tau_2(\sigma_2\sigma_1) = \sigma_2 + \sigma_1 + 1$ and the next would be $\tau_3(\sigma_3\sigma_2\sigma_1) = \sigma_1 \cdot (\sigma_2 + \sigma_3 + \sigma_2 \cdot \sigma_3)$. So rather than compute the functions τ_b , we shall explicitly express their binary values as binary functions on their respective domains of integers, modulo 2^b , ascendingly ordered from 0 to $2^b - 1$. Since 0 in the exponent (of -1) would correspond to a positive sign of the square root so we shall denote it accordingly as $+$. The value 1 (in the exponent of -1) would be respectively denoted with the negative sign $-$. So we shall represent the sign-functions τ_b for $1 \leq b \leq 14$ with (ascendingly ordered) sign sequences.

0 positive signs and 2 negative signs in the 2-term sign-sequence for τ_1 :

$--$,

2 positive signs and 2 negative signs in the 4-term sign-sequence for τ_2 :

$-++-$,

5 positive signs and 3 negative signs in the 8-term sign-sequence for τ_3 :

$+++--+-$,

10 positive signs and 6 negative signs in the 16-term sign-sequence for τ_4 :

$+ - + + + - - + + + - + + - -$,

$$++-++--++++-+- - - - + - - + - - - + + - + +,$$
[illegible]

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487 positive signs and 537 negative signs in the 1024-term sign-sequence for τ_{10} :

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1016 positive signs and 1032 negative signs in the 2048-term sign-sequence for τ_{11} :

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2045 positive signs and 2051 negative signs in the 4096-term sign-sequence for τ_{12} :

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8063 positive signs and 8321 negative signs in the 16384-term sign-sequence for τ_{14} :

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Hermes, of course, did not need to calculate all the signs which have been automatically computed. And he certainly did not need to calculate the latter 16384-sign-sequence for τ_{14} .

The algebraic integer $s^{15} = 2 \cos(2\pi/65537)$ is a root of the quadratic polynomial

$$x^2 - s^{14}x + s_{3072}^{14},$$

which, according to our notation, is

$$s^{15} = r_{15}(0) = \sqrt{(r_{14}(0)/2)^2 - r_{14}(3072)} + \frac{r_{14}(0)}{2}$$

expressible via nested radicals due to the recurrence

$$r_{b+1}(\sigma_{b+1}\sigma_b \cdots \sigma_1) = \frac{(-1)^{\sigma_{b+1}+\tau_b(\sigma)} \sqrt{\Delta_b(\sigma)} + r_b(\sigma)}{2}.$$

Hermes, on the other hand, followed his own route. The table on the RHS of this page, which appeared in his cited paper [1], had 12 of its (initial) values match the values for s^m , where $0 \leq m \leq 11$. However, after $s^{11} \approx 25.21077$ (which is η_1^{11} , according to his notation) he proceeded not with s^{12} but with $s_{2048}^{12} \approx 11.3732$ then with $s_{2048}^{13} \approx 3.9614$, $s_{10240}^{14} \approx -0.00015$, $s_{10240}^{15} = 2 \cos(65538 \pi/65537) = -2 \cos(\pi/65537) \approx -1.9999999977$, finally arriving at $e^{65538 \pi \sqrt{-1}/65537} = -e^{\pi \sqrt{-1}/65537} \approx -0.9999999989 - 0.0000479362 \sqrt{-1}$ (which is r^{32769} , according to his notation).

| | |
|----|---|
| 0 | $\eta_1 = -1$ |
| 1 | $\eta_1 = 127,5009765 \dots$ |
| 2 | $\eta_1 = -26,58292 \dots$ |
| 3 | $\eta_1 = -62,34271 \dots$ |
| 4 | $\eta_1 = -23,68614 \dots$ |
| 5 | $\eta_1 = 24,62068 \dots$ |
| 6 | $\eta_1 = 15,12077 \dots$ |
| 7 | $\eta_1 = 33,90859 \dots$ |
| 8 | $\eta_1 = 40,22580 \dots$ |
| 9 | $\eta_1 = 32,0096 \dots$ |
| 10 | $\eta_1 = 43,6329 \dots$ |
| 11 | $\eta_1 = 25,2107 \dots$ |
| 12 | $\eta_1 = 11,3732 \dots$ |
| 13 | $\eta_1 = 3,9614 \dots$ |
| 14 | $\eta_1 = -0,00015 \dots$ |
| 15 | $\eta_1 = -1,9999999978 \dots$ |
| 16 | $\eta_1 = -0,9999999989 \dots - i \cdot 0,0000479362 \dots$ |
| | $= r^{32769}.$ |

Calculations by J.G. Hermes, as presented in [1]

Acknowledgment

The reconstruction of the 65537-gon within a few days in early Spring 2026, rather than decades (in the late 19th century), became possible mainly due to the work of three wonderful assistants:

- **MathPartner** – a free, cloud-based, computer algebra system [3] – carried out significant symbolic computations.
- **Runestone** – a free streamlined text editor – efficiently performed searching, counting and replacing.
- **DeepSeek** – a free state-of-the-art large language model – generated code.

The author of this paper is grateful to all contributors to the aforementioned creations.

References

- [1] Johann Gustav Hermes. **Über die Teilung des Kreises in 65537 gleiche Teile**. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse. Göttingen, 1894: 170–186.
- [2] Nikolai Vavilov. **Fermat numbers and cyclotomy**. Seminar on the History of Mathematics, St. Petersburg, 2023, April 6, online. Available at https://www.mathnet.ru/php/seminars.phtml?presentid=38204&option_lang=eng.

- [3] Gennadi Malaschonok & Alexandr Seliverstov. **MathPartner: An Artificial Intelligence Cloud Service**. In: Arai K. (ed.) Intelligent Systems and Applications. IntelliSys 2024. Lecture Notes in Networks and Systems, vol. 1068, Springer, Cham, 2024: 182–194. Available at [doi:10.1007/978-3-031-66336-9_14](https://doi.org/10.1007/978-3-031-66336-9_14).