# A few regular polygons and Della Dumbaugh

Semjon Adlaj

April 25, 2025

## Carl Gauss and Della Dumbaugh

Carl Gauss was the "hero" of A. Gleason's paper "Angle Trisection, the Heptagona, and the Triskaidecagon" [6],[1] whereas the "hero" of our paper "A few regular polygons and Della Dumbaugh" is Della Dumbaugh. Unlike Gauss she has constructed nothing but unfortunately for the American Mathematical Monthly she became its editor and, unfortunately for her, she was misled to believe that she was either a Mathematician or a historian of Mathematics, whereas she is neither.[2] A historian of Mathematics (but not Della Dumbaugh), although need not necessarily personally contribute to Mathematics, would, of course, distinguish an explicit construction of a heptagon and a triskaidecagon, via trisection, from a mere statement concerning the possibility of constructing the hendecagon, via quintisection. The latter statement, made in [6], was not followed by an "offer for the reader to carry on the details" of constructing an hendecagon, via quintisection, unlike the construction of an enneakaidecagon,[3] via "two trisections", which "details were left to the reader!"[4] Yet, Della Dumbaugh told us in a letter that there is "little in this" (hendecagon construction via quintisection) "that has not been known for 35+ years."[5]

## Calculating the minimal polynomial of the algebraic integer $2\cos\left(\frac{2\pi}{2n+1}\right)$

Let $S^n$ denote the set of $n$ natural numbers from 1 to $n$ and let $S^n_k$ denote its $k$-subset, that is, a subset with $k$ many elements in it. The coefficients of a given monic polynomial $p$, which degree is $n$, are expressible, via Vieta's formulas, as symmetric functions of its $n$ roots $r_i$, $1 \le i \le n$:

$$p(x) = \prod_{i \in S^n} (x - r_i) = \sum_{k=0}^{n} c_k\, x^k,\ c_k = (-1)^{n-k}\sigma^{n-k},\ \sigma^k = \sum_{S^n_k \subseteq S^n} \prod_{i \in S^n_k} r_i.$$

Note that the sum in Vieta's formula for the "$k^{\text{th}}$ elementary symmetric polynomial" $\sigma^k$ contains $n!/k!/(n-k)!$ terms since there are that many subsets $S^n_k$ of $S^n$, so the formula extends to calculating the "zeroth elementary symmetric polynomial" $\sigma_0$ as the sum containing a single term which is the empty product. Thus, $\sigma^0 = 1$ and $c_n = (-1)^0\sigma^0 = 1$, as it ought to be.

[1] That paper was published in 1988 by the American Mathematical Monthly which editor then was Herbert Saul Wilf (June 13, 1931 – January 7, 2012).

[2] Actually, Della Dumbaugh is a podcaster who has unveiled herself in her own words which were (unfortunately) published

[3] The names for the polygons are based on combining a Greek-derived numerical prefix with the suffix "gon".

[4] The heptatriacontagon would be the "next" polygon to be constructed via trisections.

[5] Apparently, she "knew it" since she went to college and left us pondering on what was the "little in this" that has not been known to her.

For every positive integer $n$ denote with $p^n$ the monic polynomial, of degree $n$, which roots are

$$r_k^n = 2\cos\left(\frac{2\pi k}{2n+1}\right) = e^{2\pi k\sqrt{-1}/(2n+1)} + e^{-2\pi k\sqrt{-1}/(2n+1)},\ 1 \le k \le n,$$

and let $c_k^n$ denote its coefficients, that is,

$$p^n(x) = \sum_{k=0}^{n} c_k^n\, x^k.$$

The coefficients of $p^{n+1}$ might efficiently be calculated via an induction (on the degree $n$) as

$$c_k^{n+1} = (-1)^{n-k} c_k^n + \frac{1-(-1)^{n-k}}{2}\, c_{k-1}^n = \begin{cases} c_k^n & \text{if } n-k \text{ is even,} \\ c_{k-1}^n - c_k^n & \text{if } n-k \text{ is odd.} \end{cases} \tag{1}$$

Extending the range of "possible" values of $n$ to include zero as a "base value", we set an "induction base" as

$$c_k^0 = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \tag{2}$$

In particular, all coefficients $c_k^n$ (for all natural numbers $n$ and all integers $k$) are integers,[6] and so $r_1^n$ is an algebraic integer for every natural numbers $n$.[7]

Here is a list of the polynomials $p^n$ for $1 \le n \le 13$:[8]

$$p^1(x) = x + 1,$$

$$\begin{aligned}
p^2(x) &= x^2 + x - 1, \\
p^3(x) &= x^3 + x^2 - 2x - 1, \\
p^4(x) &= x^4 + x^3 - 3x^2 - 2x + 1, \\
p^5(x) &= x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1, \\
p^6(x) &= x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1, \\
p^7(x) &= x^7 + x^6 - 6x^5 - 5x^4 + 10x^3 + 6x^2 - 4x - 1, \\
p^8(x) &= x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1, \\
p^9(x) &= x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1, \\
p^{10}(x) &= x^{10} + x^9 - 9x^8 - 8x^7 + 28x^6 + 21x^5 - 35x^4 - 20x^3 + 15x^2 + 5x - 1, \\
p^{11}(x) &= x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1, \\
p^{12}(x) &= x^{12} + x^{11} - 11x^{10} - 10x^9 + 45x^8 + 36x^7 - 84x^6 - 56x^5 + 70x^4 + 35x^3 - 21x^2 - 6x + 1,
\end{aligned}$$

[6] Including the vanishing coefficients $c_k^n$, where either $k > n$ or $k < 0$.

[7] The minimal polynomial of $r_1^n$ over $\mathbb{Q}$ need not, of course, coincide with $p^n$ but it must evidently be a (monic) polynomial divisor of $p^n$ with integer coefficients. In fact, we shall soon see that each polynomial $p^n$ factors into a product of minimal polynomials which $n$ roots in total are the algebraic integers $r_k^n$, $1 \le k \le n$.

[8] The coefficients of these polynomials, along with the coefficient $c_0^0$, listed in an ascending order of the exponents, give rise to a sequence yet to be included in the "On-Line Encyclopedia of Integer Sequences (OEIS)": {**1**, **1**, **1**, -1, **1**, **1**, -1, -2, **1**, **1**, 1, -2, -3, **1**, **1**, 1, 3, -3, -4, **1**, **1**, -1, 3, 6, -4, -5, **1**, **1**, -1, -4, 6, 10, -5, -6, **1**, **1**, 1, -4, -10, 10, 15, -6, -7, **1**, **1**, 1, 5, -10, -20, 15, 21, -7, -8, **1**, **1**, -1, 5, 15, -20, -35, 21, 28, -8, -9, **1**, **1**, -1, -6, 15, 35, -35, -56, 28, 36, -9, -10, **1**, **1**, 1, -6, -21, 35, 70, -56, -84, 36, 45, -10, -11, **1**, **1**, 1, 7, -21, -56, 70, 126, -84, -120, 45, 55, -11, -12, **1**, **1**...}.

$$p^{13}(x) = x^{13}+x^{12}-12x^{11}-11x^{10}+55x^9+45x^8-120x^7-84x^6+126x^5+70x^4-56x^3-21x^2+7x+1.$$

We might, as well, write down an explicit formula

$$p^n(x) = \sum_{k=0}^{n} c_k^n\, x^k,\ \ c_k^n = (-1)^m \frac{(m+k)!}{k!\, m!}, \tag{3}$$

$$m = \frac{n-k}{2} + \frac{(-1)^{n-k}-1}{4} = \begin{cases} (n-k)/2 & \text{if } n-k \text{ is even,} \\ (n-1-k)/2 & \text{if } n-k \text{ is odd,} \end{cases}$$

according to which the domain for the coefficients $c_k^n$ might be extended to include all integers $k$ with the linear recurrence relation (1) still satisfied on the extended domain. We shall note, however, that the coefficients $c_k^n$, which did and still do vanish whenever $k > n$, would now, according to their formula (3), generate, for each integer $n$, a sequence $S_n = \{c_{-1}^n,\, c_{-2}^n,\, c_{-3}^n \ldots\}$.[9]

Note that $p^n$ is divisible by $p^m$ if and only if $2n+1$ is divisible by $2m+1$ and so $p^n$ is the minimal polynomial of $r_1^n$, over $\mathbb{Q}$, if and only if $2n+1$ is (an odd) prime. Denoting with $p_n$ the minimal (monic) polynomial of $r_1^n$, over $\mathbb{Q}$, we must have

$$p^n(x) = \prod_{(2m+1)|(2n+1)} p_m(x) \tag{4}$$

upon further assigning the (exceptional) identity (for the case of $n=0$) $p_0(x) = p^0(x) \equiv 1$.

We might thus proceed to factor each polynomial $p^n$ into a product of minimal polynomials, over $\mathbb{Q}$, for $1 \leq n \leq 13$:

$$\begin{gathered}
p^1(x) = p_1(x) = x+1, \\
p^2(x) = p_2(x) = x^2+x-1, \\
p^3(x) = p_3(x) = x^3+x^2-2x-1, \\
p^4(x) = p_1(x)p_4(x) = (x+1)(x^3-3x+1), \\
p^5(x) = p_5(x) = x^5+x^4-4x^3-3x^2+3x+1, \\
p^6(x) = p_6(x) = x^6+x^5-5x^4-4x^3+6x^2+3x-1, \\
p^7(x) = p_1(x)p_2(x)p_7(x) = (x+1)(x^2+x-1)(x^4-x^3-4x^2+4x+1), \\
p^8(x) = p_8(x) = x^8+x^7-7x^6-6x^5+15x^4+10x^3-10x^2-4x+1, \\
p^9(x) = p_9(x) = x^9+x^8-8x^7-7x^6+21x^5+15x^4-20x^3-10x^2+5x+1, \\
p^{10}(x) = p_1(x)p_3(x)p_{10}(x) = (x+1)(x^3+x^2-2x-1)(x^6-x^5-6x^4+6x^3+8x^2-8x+1),
\end{gathered}$$

[9] Note that in the case of negative values of $k$, the assumption of the vanishing coefficients $c_k^0$, as given in (2), no longer agrees with formula (3) which, as we remarked earlier, would generate a family of sequences $S_n$. We might, in particular, list 13 terms of the sequence $S_0 = \{1,1,2,3,6,10,20,35,70,126,252,462,924\ldots\}$. Remarkably, however, that recurrence relation (1), upon alternatively extending it to include $S_0$ as a "base sequence", would still be consistent with the identity $c_{-1}^n = 0$, for all $n > 0$, which we had earlier upon assuming (2). Moreover, the first $n$ terms of each sequence $S_n$ are guaranteed to vanish (without requiring the terms $c_k^0$ to vanish for negative $k$)! Let us demonstrate this observation with listing 13 terms of each of the following successive sequences: $S_1 = \{0,1,1,3,4,10,15,35,56,126,210,462,792\ldots\}$, $S_2 = \{0,0,1,1,4,5,15,21,56,84,210,330,792\ldots\}$, $S_3 = \{0,0,0,1,1,5,6,21,28,84,120,330,495\ldots\}$, $S_4 = \{0,0,0,0,1,1,6,7,28,36,120,165,495\ldots\}$, $S_5 = \{0,0,0,0,0,1,1,7,8,36,45,165,220\ldots\}$, $S_6 = \{0,0,0,0,0,0,1,1,8,9,45,55,220\ldots\}$, $S_7 = \{0,0,0,0,0,0,0,1,1,9,10,55,66\ldots\}$, $S_8 = \{0,0,0,0,0,0,0,0,1,1,10,11,66\ldots\}$, $S_9 = \{0,0,0,0,0,0,0,0,0,1,1,11,12\ldots\}$, $S_{10} = \{0,0,0,0,0,0,0,0,0,0,1,1,12\ldots\}$, $S_{11} = \{0,0,0,0,0,0,0,0,0,0,0,1,1\ldots\}$.

$$p^{11}(x) = p_{11}(x) = x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1,$$
$$p^{12}(x) = p_2(x)p_{12}(x) = (x^2 + x - 1)(x^{10} - 10x^8 + 35x^6 + x^5 - 50x^4 - 5x^3 + 25x^2 + 5x - 1),$$
$$p^{13}(x) = p_1(x)p_4(x)p_{13}(x) = (x + 1)(x^3 - 3x + 1)(x^9 - 9x^7 + 27x^5 - 30x^3 + 9x + 1).$$

Denoting the degree of a polynomial $p$ with $d(p)$, we might (recursively) deduce from expression (4), for $p^n$, a formula for the degree of the minimal polynomial $p_n$, that is,

$$d(p_n) = d(p^n) - \sum_{(2m+1)\in T^n} d(p_m) = n - \tau(n) = \pi(n),$$ [10]

where $T^n$ denotes the set of proper divisors of $2n+1$, $\tau(n)$ is the number of elements of $S^n$ which share a non-trivial divisor with $2n+1$ and $\pi(n)$ is the number of elements of $S^n$ which are coprime with $2n + 1$.[11] For $n \neq 0$, $\pi(n) = \phi(2n + 1)/2$, where $\phi(\cdot)$ is the Euler's Totient function, which means that our formula agrees with the formula derived in [7]. Note, however, that $\phi(1) = 1$ which means that the agreement ceases in the exceptional case with $n = 0$, as was noted in [8].

Evidently, if $2n + 1$ is (an odd) prime then $p_n = p^n$ and (consequently) $d(p_n) = d(p^n) = n$.

## Polynomial factorization towards the construction of regular polygons

The degree of any minimal polynomial $p_n$ matches the degree of its splitting field, over $\mathbb{Q}$, since any field extension of $\mathbb{Q}$ containing a root of $p_n$ necessarily contains all of its roots.[12] No intermediate field extensions between $\mathbb{Q}$ and the splitting field for $p_n$ exist whenever both $n$ and $2n + 1$ are primes.[13]

An intermediate factorization of a polynomials $p_n$, where $2n+1$ is composite, might be attained by adjoining to $\mathbb{Q}$ a root of $p_m$, where $2m + 1$ is a proper divisor of $2n + 1$. In particular,

$$p_7(x) = x^4 - x^3 - 4x^2 + 4x + 1 = \left(x^2 + r_1^2\, x + r_2^2 - 1\right)\left(x^2 + r_2^2\, x + r_1^2 - 1\right),$$

$$p_{10}(x) = x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 =$$
$$= \left(x^2 + r_1^3\, x + (r_1^3)^2 - 3\right)\left(x^2 + r_2^3\, x + (r_2^3)^2 - 3\right)\left(x^2 + r_3^3\, x + (r_3^3)^2 - 3\right),$$
$$p_{12} = x^{10} - 10x^8 + 35x^6 + x^5 - 50x^4 - 5x^3 + 25x^2 + 5x - 1 = (x^5 - 5\,x^3 + 5\,x - r_1^2)(x^5 - 5\,x^3 + 5\,x - r_2^2),$$
$$p_{13} = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x + 1 = \left(x^3 - 3\,x - r_1^4\right)\left(x^3 - 3\,x - r_2^4\right)\left(x^3 - 3\,x - r_3^4\right).$$

An intermediate factorization of a polynomials $p_n$, where $2n + 1$ is prime, is attainable as long as $n$ is composite.

$$p_6(x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1 =$$
$$= \left(x^3 + (u_{13}^+ - 1)\, x^2 - x - u_{13}^+\right)\left(x^3 + (u_{13}^- - 1)\, x^2 - x - u_{13}^-\right),$$
$$p_8(x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 =$$

[10] Thus, the formulas for $d(p_n)$ give rise to yet another sequence, which upon choosing an offset 1, would coincide with the sequence A072451 in OEIS: $1, 2, 3, 3, 5, 6, 4, 8, 9, 6, 11, 10, 9 \ldots$.

[11] We must point out here that our chosen notation is reminiscent of the widespread notational convention (but does not coincide with it), where $\pi(n)$ is the number of primes in $S^n$. Note, as well, that the (non-prime) number 1 is coprime with all natural numbers. And since the number of elements of the empty set is 0 (and so must be the number of elements of the empty set which are coprime with 1) we must have $\pi(0) = 0$, being the degree of the constant polynomial 1.

[12] In other words, adjoining a root of $p_n$ to $\mathbb{Q}$ is a Galois extension of $\mathbb{Q}$.

[13] The corresponding sequence is A079148 in OEIS which is the subsequence of primes $p$, where "$p - 1$ has at most 2 prime factors". These are the primes $3, 5, 7, 11, 23 \ldots$

$$= \left(x^4 + u_{17}^+ x^3 - (u_{17}^- + 1)\, x^2 + (2\, u_{17}^- + 1)\, x - 1\right) \left(x^4 + u_{17}^- x^3 - (u_{17}^+ + 1)\, x^2 + (2\, u_{17}^+ + 1)\, x - 1\right) =$$

$$\left(x^2 + \frac{u_{17}^- + v_{17}^-}{2}\, x - \frac{u_{17}^+ + v_{17}^+}{2}\right) \left(x^2 + \frac{u_{17}^- - v_{17}^-}{2}\, x - \frac{u_{17}^+ - v_{17}^+}{2}\right) \cdot$$

$$\cdot \left(x^2 + \frac{u_{17}^+ + v_{17}^+}{2}\, x - \frac{u_{17}^- - v_{17}^-}{2}\right) \left(x^2 + \frac{u_{17}^+ - v_{17}^+}{2}\, x - \frac{u_{17}^- + v_{17}^-}{2}\right),$$

$$p_9(x) = x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1 =$$

$$(x^3 + (\beta^3 + 1)x^2 + \beta^2 x + \beta^1)(x^3 + (\beta^1 + 1)x^2 + \beta^3 x + \beta^2)(x^3 + (\beta^2 + 1)x^2 + \beta^1 x + \beta^3),$$

where

$$u_{13}^\pm := \frac{3 \pm \sqrt{13}}{2},\ u_{17}^\pm := \frac{1 \pm \sqrt{17}}{2},\ v_{17}^\pm := \sqrt{\pm\sqrt{17}\, u_{17}^\pm}\,,$$ [14]

$$\beta^s = -\frac{e^{2\pi s\sqrt{-1}/3}\gamma_- + e^{-2\pi s\sqrt{-1}/3}\gamma_+ + 2}{3},\ \gamma_\pm = \left(\frac{19\,(7 \pm 3\sqrt{-3}\,)}{2}\right)^{1/3}.$$

A construction of triskaidecagon (via angle trisection) based upon factoring $p^6$ into a product of two (conjugate) cubic polynomials was offered in [6]. The same source indicates that Gauss constructed the 17-gon in 1796 and states (as we mentioned earlier) that the construction of the 19-gon requires two trisections which “details were left to the reader”.

No intermediate factorization exists for of $p_n$ if both $2n + 1$ and $n$ are primes. Thus, for any given field extension of $\mathbb{Q}$, a polynomial such as $p_5$ or $p_{11}$ either remains irreducible or (exclusively and completely) splits into linear factors. Constructing the roots of these solvable polynomials ($p_5$ and $p_{11}$) enable a construction of a hendecagon via quintisection and a construction of a 23-gon via 5 and 11 angle section, respectively.

## Expressing in radicals the roots of the solvable polynomials $p^5$ and $p^{11}$

In order to solve the polynomial $p^5$ we introduce the Lagrange resolvent as

$$\rho(\eta) := r_1^5\, \eta + r_2^5\, \eta^2 + r_4^5\, \eta^3 + r_3^5\, \eta^4 + r_5^5\,,$$ [15]

where $\eta$ is a fifth, not necessarily primitive, root of unity.

Evidently, the value of the Lagrange resolvent does depend on the chosen fifth root of unity. Subsequently, for the sake of brevity, we shall denote $\rho(\xi_5^k)$, where $\xi_5 := e^{2\,\pi\,\sqrt{-1}/5}$, by $\rho_k$. So, in particular, $\rho_0 = \rho(1) = -1$, that is, the sum of the roots of $p^5$.

Now note that

$$5\, r_5^5 = \rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4,$$

so the quintuples of the roots of $p^5$ are expressible as sums of fifth roots of the five values $\rho_k^5$ all of which might be calculated once we calculate

$$\rho_1^5 = 90\, \xi_5^4 - 20\, \xi_5^3 + 255\, \xi_5^2 - 130\, \xi_5 - 196$$

and proceed to replace $\xi_5$ (on the right-hand side) by the fifth roots of unity $\xi_5^k$,[16] including the trivial root for which $k = 0$, corresponding to the value $\rho_0^5 = -1$ (as it ought to be since we already

[14] Only the upper (exclusively) or the ower sign is meant to be taken on both sides of an equation.

[15] There is no typing error here, as might naively be perceived, that is, we ought not swap $r_4^5$ with $r_3^5$ without simultaneously swapping $\eta^3$ with $\eta^4$.

[16] For $k \in \{1, 2, 3, 4\}$, the set of the four values $\{k,\ 2k,\ 3k,\ 4k\}$ is the same (mod 5) as the set $\{1, 2, 3, 4\}$.

know that $\rho_0 = -1$). The four values, corresponding to four primitive fifth roots of unity, might be paired as two complex conjugate pairs $u_\pm$, $v_\pm$, where

$$\rho_1^5 = u_+,\ \rho_2^5 = v_+,\ \rho_3^5 = v_-,\ \rho_4^5 = u_-,$$

$$u_\pm = -\frac{11\left(89 + 25\sqrt{5} \pm 5\sqrt{178\sqrt{5} - 410}\right)}{4},\ v_\pm = -\frac{11\left(89 - 25\sqrt{5} \pm 5\sqrt{-178\sqrt{5} - 410}\right)}{4},$$

and so $\rho_1^5$, $\rho_2^5$, $\rho_3^5$ and $\rho_4^5$ are the four roots of the polynomial

$$\left(x^2 + \frac{11(89 + 25\sqrt{5})}{2}x + 11^5\right)\left(x^2 + \frac{11(89 - 25\sqrt{5})}{2}x + 11^5\right) =$$

$$= x^4 + 89 \cdot 11\, x^3 + 13 \cdot 33^3\, x^2 + 89 \cdot 11^6\, x + 11^{10}.$$

Formidable attempts for constructing a regular hendecagon were ignited at "the Mathematics Stack Exchange" on July 31, 2016 with "Constructing the 11-gon by splitting an angle in five" and further discussed in "Simplifying a radical-trigonometric expression for the hendecagon angle" till August 23, 2023. However, the construction on June 13, 2022 in "Radical representation of $\cos\frac{2\pi}{11}$" came even closer to the construction which we publicized on December 14, 2024.[17]

The preceding calculation of the roots of $p^5$ might be generalized to calculating the roots of $p^n$ whenever both $n$ and $2n+1$ are primes. We would then put

$$\rho_k := \rho(\xi_n^k),\ \rho(\eta) := \sum_{k=0}^{n-1} r_{2^k}^n \eta^k,\ \xi_n := e^{2\pi\sqrt{-1}/n},$$ [18]

and observe that

$$n\, r_n^n = \sum_{k=0}^{n-1} \rho_k,$$

so the $n$-multiples of the roots of $p_n$ are expressible as sums of $n^{\text{th}}$ roots of the $n-1$ values $\rho_k^n$, $1 \le k \le n-1$ (that is, $(n-1)/2$ complex conjugate pairs), in addition to the value $\rho_0 = -1$.[19]

In order to derive an explicit expression for $\rho_k^n$ we shall denote by $P_n$ the set of all integer partitions of $n$.[20] To each partition $P_n^j \in P_n$ we might associate an $n$-tuple of integers $(j_1, \ldots, j_n)$, adhering to the conditions

$$\sum_{i=1}^n j_i = n,\ n \ge j_1 \ge \cdots \ge j_n \ge 0.$$

[17] Only one last simplification was missing, thereby sending others on a wrong track of activating "Mathematica ToRadicals", leading to a horrible radical expression. Amuzingly, the Wolfram MathWorld "Trigonometry Angles" site (Last Updated: Fri Mar 28 2025) indicates that the case of the prime 23 (which we are about to deal with here after Della Dumbaugh failed to "postpone" it) "is a very difficult case and takes a long time to calculate". Yet, things are far worse for "Wolfram Alpha" than that site is willing to admit as it fails to indicate the size of the output after that "long time" is spent.

[18] When both $n$ and $2n+1$ are primes, the multiplicative group (of order $2n$) of the field of $2n+1$ elements contains a maximal number $(n-1)$ of generators. The element 2, in particular, is either a generator or a square thereof, and so is guaranteed to generate no less than all $n$ distinct squares, including the identity element. Thus and furthermore, 2 might be replaced by any of its powers $2^k$, as long as $k$ is not a multiple of $n$. We must keep in mind, however, that such replacement would induce a cyclic permutation on the resolvents $\rho_k$, $1 \le k \le n-1$, corresponding to primitive roots, while $\rho_0$ remains fixed. And the fact that the length of the cycle $(n-1)$ coincides with the number of generatos (or their squares) serves to remind us that the permutation (of $\rho_k$) is trivial if 2 is replaced with $-2$.

[19] All other $n^{\text{th}}$ roots of $\rho_0^n = -1$ are excluded.

[20] We have succumbed to the common use of the term "integer partitions" which admittedly is better replaced by "natural partitions" in order to emphasize that negative integers are excluded here.

The alternating group $A_{n-1}$ (which is the subgroup of even permutations of the corresponding symmetric group) acts on the (ordered) $n-1$-tuple $(j_2, \ldots, j_n)$ (via permuting the indices). So if we denote by $Q_n^j$ the orbit of $P_n^j$ under the said action of $A_{n-1}$ and if $(k_1, \ldots, k_n) \in Q_n^j$ then $k_1 = j_1$, that is, $j_1$ remains "fixed in its position".[21]

Let $s_k$ denote a sum of of $k$ (strictly) positive terms and consider the "total" sum of $n$ (not necessarily distinct) powers of 2

$$s_n = s_n(k_1, \cdots, k_n) := \sum_{i=1}^{n} k_i 2^i$$

for which we might construct a monomial

$$x^s, \ s = s_n - 2s_k \pmod{2n+1}, \tag{5}$$

corresponding to a "partial" sum $s_k$ of (exactly) $k$ summands (powers of 2). There are $2^{n-1}$ such (not necessarily distinct) monomials as $k$ ranges from 0 to $(n-1)/2$, inclusively, and so we might define the polynomial $b(x) = b(x, k_1, \cdots, k_n)$ as the sum of these monomials.[22]

We are now ready to write down the sought for expression as

$$\rho_k^n = \sum_{P_n^j \in P_n} \frac{n!}{\alpha(j_1)} \prod_{i=1}^{n} \frac{1}{j_i!} \sum_{\{k_1,\ldots,k_n\} \in Q_n^j} \left((2n+1)b(0) - 2^{n-1}\right) \xi_n^{km}, \ m := \sum_{i=1}^{n-1} k_i i \pmod{n},$$

where $\alpha(j_1)$ denotes the number of occurrences of $j_1$ in $(j_1, \ldots, j_n)$.[23]

The latter expression, with $n = 11$, was used in [2] for constructing an icositrigon (that so much troubled "Wolfram Alpha" while Della Dumbaugh "was not losing any sleep" over it) for which

$$\rho_1^{11} = -5332019\,\xi_{11}^{10} - 1781164\,\xi_{11}^{9} - 2099438\,\xi_{11}^{8} - 5853705\,\xi_{11}^{7} + 290906\,\xi_{11}^{6} - 4039948\,\xi_{11}^{5} +$$

$$4283752\,\xi_{11}^{4} + 15151367\,\xi_{11}^{3} + 12996313\,\xi_{11}^{2} - 18426793\,\xi_{11} + 4810728.$$

It was not used, with $n = 3$, for constructing a heptagon for which

$$\rho_1^3 = -12\,\xi_3^2 + 9\,\xi_3 + 2 = \frac{7(1 + 3\sqrt{-3}\,)}{2}.$$ [24]

## The Legendre symbol of a given order and its applications

Let $\mathbb{F}_p$ denote a prime field of order $p$ and let $\xi_k$ denote a primitive $k^{\text{th}}$ root of unity, where $k$ is a divisor of $p-1$. Introduce the Legendre symbol of order $k$, which we shall denote as $L_k^p$, as a function on $\mathbb{F}_p$ which range includes zero along with the powers of $\xi_k$. The Legendre symbol of order $k$ would then map zero to zero and is determined by a homomorphism mapping a generator

[21] Recall that the number of elements of the orbit $Q_n^j$ might be obtained by dividing the order of the group $A_{n-1}$ by the order of its subgroup which stabilizes $(j_2, \ldots, j_n)$.

[22] Note that $b(1) = 2^{n-1}$ (by construction).

[23] That number is no less than one and would evidently coincide with the number of occurrences of $j_1$ in $(j_1 = k_1, \ldots, k_n) \in Q_n^j$.

[24] The case of the heptagon is exceptional since $A_2$ is the trivial group and its action must be "augmented" here with the action of the "full" symmetric group (with two elements).

$g$ of the multiplicative group of $\mathbb{F}_p$ to $\xi_k$ which generates a group isomorphic with the additive group $\mathbb{Z}_k$.[25] Explicitly, we have

$$L_k^p(0) = 0, \ L_k^p(g^l) = \xi_k^l.$$

Evidently, the Legendre symbol of order $k$ depends on the choice of the generator $g$, unless $k$ is either one or two. In fact, the widely accepted definition of the Legendre symbol (which order is not usually specified) agrees with our definition of the Legendre symbol of order 2.

We note that the value $b(0)$ which appeared in the preceding section might be expressed via a sum of Legendre symbols of order 1, that is,

$$b(0) = \sum_s 1 - L_1^{2n+1}(s),$$

where $s$ ranges over all the sums defined in (5),[26] and so

$$(2n+1)b(0) - 2^{n-1} = 2n\lambda_1,$$

$$\lambda_1 = \lambda_1(k_1, \ldots, k_n) := \sum_s \left(1 - L_1^{2n+1}(s) - \frac{L_1^{2n+1}(s)}{2n}\right).$$

The formula for calculating the $n^{\text{th}}$ power of the Lagrange resolvent when both $n$ and $2n+1$ are primes might be extended to instances when both $n$ and $4n+1$ are primes and to instances when both $n$ and $6n+1$ are primes.[27] The minimal polynomial over $\mathbb{Q}$ would respectively be $p^n$, $p^{2n}$ and $p^{3n}$. The polynomial $p^{2n}$ ($p^{3n}$) might then be factored over a quadratic (cubic) extension of $\mathbb{Q}$ into a product of two (three) polynomials of degree $n$, so we have

$$p^{2n} = q_+^n(x)\, q_-^n(x), \ p^{3n} = q_0^n(x)\, q_1^n(x)\, q_2^n(x),$$

where we might further assume that the roots of $q_\pm^n$ are those $r_j^{2n}$ for which $L_2^{4n+1}(j) = \pm 1$, whereas the roots of $q_i^n$ are those $r_j^{3n}$ for which $L_3^{6n+1}(j) = \xi_3^i$.[28] The sum of the roots of $q_\pm^n$ is $(\pm\sqrt{4n+1} - 1)/2$, whereas the sum of the roots of $q_i^n$ is $(\sqrt{6n+1}\,(\xi_3^i u + \xi_3^{-i} u^{-1}) - 1)/3$, where $u$ is a cube root of unit modulus.[29]

We would then have

$$\rho_k^n = \sum_{P_n^j \in P_n} \frac{2n}{\alpha(j_1)} \prod_{i=1}^n \frac{i}{j_i!} \sum_{\{k_1,\ldots,k_n\} \in Q_n^j} \lambda\left(k_1, \ldots, k_n\right) \xi_n^{km}, \tag{6}$$

[25]Note that we did not require $k$ to be a prime.

[26]Recall that there are $2^{n-1}$ such sums so the values $s$ might be repeated.

[27]These are, of course, not mutually exclusive instances. For example, $13 = 4\cdot 3+1 = 6\cdot 2+1$. Amusingly, for a prime $n = 3$, we have the primes $2n+1 = 7$, $2\cdot 2n+1 = 13$, $2\cdot 3n+1 = 19$, $2\cdot 5n+1 = 31$, $2\cdot 7n+1 = 43$, $2\cdot 11n+1 = 67$, $2\cdot 13n+1 = 79$ and $2\cdot 17n+1 = 103$.

[28]Note that $-1$ is a square in $\mathbb{F}_{4n+1}$ and is a cube in $\mathbb{F}_{6n+1}$

[29]The two sums $(\pm\sqrt{4n+1}-1)/2$ are the two roots of the quadratic polynomial $f_n(x) := x^2+x-n$. The three sums $(\sqrt{6n+1}\,(\xi_3^i u + \xi_3^{-i} u^{-1}) - 1)/3$ are, in turn, the roots of the cubic polynomial

$$g_n(x) := x^3 + x^2 - 2n\,x - \frac{(6n+1)\,p_n - 4n^2}{3},$$

where $p_n$ is the number of partitions of $1 \in \mathbb{F}_{6n+1}$ into a sum of two non cubes which product is a cube. Note that $p_1 = 1$ (since $3+5$ is the only partition of $1 \in \mathbb{F}_7$ meeting the requirements) and so $g_1(x) = x^3+x^2-2\,x-1 = p^3(x)$. A short list of several more values would be $p_2 = 1$, $p_3 = 3$, $p_5 = 4$, $p_6 = 3$, $p_7 = 4$, $p_{10} = 7$, $p_{11} = 7$, $p_{12} = 9$, $p_{13} = 7$. It corresponds to the cubic polynomials $g_2(x) = x^3+x^2-4\,x+1$, $g_3(x) = x^3+x^2-6\,x-7$, $g_5(x) = x^3+x^2-10\,x-8$, $g_6(x) = x^3+x^2-12\,x+11$, $g_7(x) = x^3+x^2-14\,x+8$, $g_{10}(x) = x^3+x^2-20\,x-9$, $g_{11}(x) = x^3+x^2-22\,x+5$, $g_{12}(x) = x^3+x^2-24\,x-27$, $g_{13}(x) = x^3+x^2-26\,x+41$.

$$m := \sum_{i=1}^{n-1} k_i i \pmod n,\ \lambda(k_1,\dots,k_n) =$$

$$\begin{cases} \lambda_1 = \sum_s \left(1 - L_1^{2n+1}(s) - \frac{L_1^{2n+1}(s)}{2n}\right) & \text{if } n \text{ and } 2n+1 \text{ are primes,} \\ \lambda_2 = \sum_s \left(1 - L_1^{4n+1}(s) - \frac{L_1^{4n+1}(s) - \sqrt{4n+1}\, L_2^{4n+1}(s)}{4n}\right) & \text{if } n \text{ and } 4n+1 \text{ are primes,} \\ \lambda_3 = \sum_s \left(1 - L_1^{6n+1}(s) - \frac{L_1^{6n+1}(s) - \sqrt{6n+1}\left(u\, L_3^{6n+1}(s) + (u\,(L_3^{6n+1}(s))^{-1}\right)}{6n}\right) & \text{if } n \text{ and } 6n+1 \text{ are primes,} \end{cases}$$

where $s = s_n - 2\, s_k$ is an element from the domain of the corresponding Legendre symbol, so it must be regarded accordingly as an element from $\mathbb{F}_{2n+1}$, $\mathbb{F}_{4n+1}$ or $\mathbb{F}_{6n+1}$, whereas

$$s_n = s_n(k_1, \cdots, k_n) := \sum_{i=1}^{n} k_i\, g^i,$$

with $g$ accordingly is either a generator,[30] a square of a generator or a cube of a generator of the corresponding multiplicative group of either $\mathbb{F}_{2n+1}$, $\mathbb{F}_{4n+1}$ or $\mathbb{F}_{6n+1}$.

The case of the prime $13 = 6 \cdot 2 + 1 = 4 \cdot 3 + 1$ must be dealt with separately.[31] So we might write

$$2\, r_1^6 = \rho_0 + \rho_1,$$ [32]

where

$$\rho_0 = r_1^6 + r_5^6 = \frac{\sqrt{13}}{3}\left(\sqrt[3]{\frac{-5 + 3\sqrt{-3}}{2\sqrt{13}}} + \sqrt[3]{\frac{-5 - 3\sqrt{-3}}{2\sqrt{13}}}\right) - \frac{1}{3},$$ [33]

and $\rho_1^2$ is the discriminant of the quadratic polynomial $q_0^2(x) = x^2 - \rho_0\, x + 2\, \lambda_3(1,1)$ which we might explicitly calculate as

$$\rho_1^2 = 4\left(\lambda_3\,(2,0) - \lambda_3\,(1,1)\right) = \frac{1}{3}\left(\sqrt[3]{\frac{13(131 + 15\sqrt{-3})}{2}} + \sqrt[3]{\frac{13(131 - 15\sqrt{-3})}{2}} + 13\right).$$

Or we might, alternatively, write

$$3\, r_1^6 = \rho_0 + \rho_1 + \rho_2,$$

where

$$\rho_0 = r_1^6 + r_3^6 + r_4^6 = \frac{\sqrt{13} - 1}{2}$$

and the complex conjugate pair $13 + \sqrt{13}\,(3\,\xi_3^{\pm} - 1)$ must coincide with the pair $\rho_1^3$, $\rho_2^3$.

The root $r_1^6$ of $p^6(x) = q_+^3(x)\, q_-^3(x) = q_0^2(x)\, q_1^2(x)\, q_2^2(x)$ might also be calculated as the only root of the greatest common divisor of $q_+^3$ and $q_0^2$, and thus explicitly expressed as

$$r_1^6 = \frac{1}{6}\left(\sqrt{13} - 1 + \left(\sqrt{13} + 1 + 4\,\xi_3\right)\sqrt[3]{\frac{3\,\xi_3 - 1}{\sqrt{13}}} + \left(\sqrt{13} + 1 + 4\,\xi_3^{-1}\right)\sqrt[3]{\frac{3\,\xi_3^{-1} - 1}{\sqrt{13}}}\right),$$ [33]

[30] A generator might be replaced by its square but not vice versa, that is, a generator ought not replace its square.

[31] Only one of the two one-element subsets of a two-element set would be taken in order to calculate a sum $s$ as $s_2 - 2s_1 \in \mathbb{F}_{13}$. The value $L_3^{13}(s)$, however, does not depend on the chosen value of $s_1$ since $-1$ is a cube in $\mathbb{F}_{13}$.

[32] Tacitly assuming the positivity of $\rho_1$ and so we must (consequently) have $\rho_0 - \rho_1 = r_5^6$.

[33] The cube roots here lie in the fourth and the first quadrant, respectively.

thereby once more arriving at an expression equivalent to the one which was obtained in [6]:[34]

$$r_1^6 = \frac{1}{6}\left(\sqrt{13} - 1 + 2\sqrt{26 - 2\sqrt{13}}\cos\left(\frac{1}{3}\arctan\frac{5 + 2\sqrt{13}}{3\sqrt{3}}\right)\right).$$

We shall not avoid the case of the prime $19 = 6 \cdot 3 + 1$ (which has awaited us "for 35+ years"), where we already know that

$$\rho_0 = r_8^9 + r_7^9 + r_1^9 = \frac{\sqrt[3]{(7 + 3\sqrt{-3}\,)\,19/2} + \sqrt[3]{(7 - 3\sqrt{-3}\,)19/2} - 1}{3},^{35}$$

so we calculate the complex conjugate pair $\rho_1^3$, $\rho_2^3$ as

$$\left(r_8^9\,\xi_3^{\pm 1} + r_7^9\,\xi_3^{\mp 1} + r_1^9\right)^3 = \frac{38 + \sqrt[3]{(11 \mp 21\sqrt{-3}\,)\,38^2} + \sqrt[3]{(3751 \mp 7959\sqrt{-3}\,)19/2}}{3}.^{36}$$

Now and although the task of constructing, via trisections, the "next" polygon, that is, the 37-gon has not been "assigned" to us, we shall exceed the expectations by noting that the three roots $r_1^{18}$, $r_{10}^{18}$, $r_{11}^{18}$ of $p^{18}(x) = q_+^9(x)\,q_-^9(x) = q_0^6(x)\,q_1^6(x)\,q_2^6(x)$ are the roots of the cubic polynomial which is the greatest common divisor of the polynomials $q_+^9(x) = x^9 - yx^8 - 4x^7 + (6y-3)x^6 + yx^5 + (14 - 11y)x^4 + (9-4y)x^3 + 8(y-2)x^2 + (3y-7)x - 2y + 5$ and $q_0^6(x) = x^6 - zx^5 - (z+2)x^4 + (z^2 + 6z - 6)x^3 - (z^2 - z - 2)x^2 - (2z^2 + 5z - 8)x + z^2 + 3z - 5$, where

$$y = r_1^{18} + r_4^{18} + r_{16}^{18} + r_{10}^{18} + r_3^{18} + r_{12}^{18} + r_{11}^{18} + r_7^{18} + r_9^{18} = \frac{\sqrt{37} - 1}{2}$$

is a root of $f_9(x) = x^2 + x - 9$ and

$$z = r_1^{18} + r_8^{18} + r_{10}^{18} + r_6^{18} + r_{11}^{18} + r_{14}^{18} = \frac{1}{3}\left(\sqrt[3]{\frac{37(-11 - 3\sqrt{-3}\,)}{2}} + \sqrt[3]{\frac{37(-11 + 3\sqrt{-3}\,)}{2}} - 1\right)^{35}$$

is a root of $g_6(x) = x^3 + x^2 - 12x + 11$. Explicitly, the cubic polynomial

$$(x - r_1^{18})(x - r_{10}^{18})(x - r_{11}^{18}) =$$

$$x^3 - \frac{z}{2}x^2 + \frac{z^2 + 3z - 8}{2}x - \frac{z^2}{2} - z + 2 + \frac{(-4z^2 - 15z + 16)\,x^2 + (7z^2 + 17z - 28)\,x + 3z^2 + 2z - 12}{2\sqrt{37}}$$

possesses among its three roots the root

$$r_1^{18} = \frac{\sqrt[3]{37^2\gamma + 37^3\sqrt{6^3\delta}} + \sqrt[3]{37^2\gamma - 37^3\sqrt{6^3\delta}} + \sqrt{37}(4z^2 + 15z - 16) + 37z}{222} \approx 1.97123^{37},$$

where $\gamma := \left(518 - 202\sqrt{37}\right)z^2 + \left(444 - 332\sqrt{37}\right)z + 1178\sqrt{37} - 518$ and $\delta := (\sqrt{37} - 5)\,(z^2 + 3z - 4) - 8$.

---

[34]Simultaneously arriving at an expression for $\rho_1$ as the difference

$$r_1^6 - r_5^6 = \frac{1}{3}\left(\sqrt{13} + (1 + 4\,\xi_3)\sqrt[3]{\frac{3\,\xi_3 - 1}{\sqrt{13}}} + \left(1 + 4\,\xi_3^{-1}\right)\sqrt[3]{\frac{3\,\xi_3^{-1} - 1}{\sqrt{13}}}\right),$$

which square is the, earlier calculated, discriminant $\rho_1^2$ of the polynomial $q_0^2$.

[35]This equality is better viewed as a definition of the right-hand side, where particular complex conjugate branches of the cube root (from the fourth and the first quadrant, respectively) must be taken, via the left.

[36]The value of a cube root here lies in the quadrant of its argument.

[37]The approximation given here is our "lazy" way of specifying the cube roots.

## Expressing in radicals the roots of the solvable polynomials $p^{14}$ and $p^{15}$

The minimal polynomial $p^{14}$ factors over a quadratic extension of $\mathbb{Q}$ into a product of two septic polynomials, that is,

$$p^{14}(x) = q^7_+(x)\, q^7_-(x),\ \ q^7_\pm(x) = x^7 - \frac{\pm\sqrt{29}-1}{2}\, x^6 - 3\, x^5 + \frac{\pm 5\sqrt{29}-9}{2}\, x^4 +$$

$$+\frac{\pm\sqrt{29}-11}{2}\, x^3 - \frac{\pm 5\sqrt{29}-19}{2}\, x^2 - (\pm 2\sqrt{29}-13)\, x - \frac{\pm\sqrt{29}-5}{2},$$

whereas the minimal polynomial $p^{15}$ factors over a cubic extension of $\mathbb{Q}$ into a product of three quintic polynomials, that is,

$$p_{15}(x) = q^5_0(x)\, q^5_1(x)\, q^5_2(x),\ \ q^5_i(x) = x^5 - \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\,\right)}\,\xi_3^i + \sqrt[3]{31\left(2-3\sqrt{-3}\,\right)}\,\xi_3^{-i} - 1}{3}\, x^4 +$$

$$+\frac{\sqrt[3]{31\left(2+3\sqrt{-3}\right)}\,\xi_3^{i-1} + \sqrt[3]{31\left(2-3\sqrt{-3}\right)}\,\xi_3^{1-i} - 4}{3} x^3 +$$

$$+\frac{\sqrt[3]{31\left(101+12\sqrt{-3}\,\right)}\,\xi_3^i + \sqrt[3]{31\left(101-12\sqrt{-3}\,\right)}\,\xi_3^{-i} + 8}{3}\, x^2 +$$

$$+\frac{2\left(\sqrt[3]{31\left(2+3\sqrt{-3}\right)}\,\xi_3^{i-2} + \sqrt[3]{31\left(2-3\sqrt{-3}\right)}\,\xi_2^{2-i} - 4\right)}{3}\, x - 1.$$

Applying (6) to calculating the roots of $q^7_+$ which sum to

$$\rho_0 = \frac{\sqrt{29}-1}{2},$$

we get

$$\rho_1^7 = 10885\,\xi_7^6 + 3171\,\xi_7^5 - 11039\,\xi_7^4 - \frac{40145}{2}\,\xi_7^3 + \frac{5327}{2}\,\xi_7^2 + \frac{63385}{2}\,\xi_7 - 18869 +$$

$$\sqrt{29}\,(-3605\,\xi_7^6 + 5761\,\xi_7^5 + 840\,\xi_7^4 - \frac{4137}{2}\,\xi_7^3 + \frac{6601}{2}\,\xi_7^2 - \frac{3563}{2}\,\xi_7 - 2110).$$ [38]

The latter expression might be restricted to the three vertices of the heptagon $\xi_7$, $\xi_7^2$, $\xi_7^4$:

$$\rho_1^7 = \frac{7(18879 - 1709\sqrt{29} - 6641\sqrt{-7} - 2237\sqrt{-203}\,)}{4}\,\xi_7^2 +$$

$$+\frac{7(19981 - 6411\sqrt{29} - 8845\sqrt{-7} + 439\sqrt{-203}\,)}{4}\,\xi_7 - \frac{84071}{2} - 8557\sqrt{29} - 10962\sqrt{-7} + \frac{4445\sqrt{-203}}{2},$$

which would, upon cyclically permuting the (ordered) three verices, generate the three values $\rho_1^7$, $\rho_2^7$, $\rho_4^7$, thereby enabling the construction of the icosienneagon via 3 and 7 angle section [3].

Applying (6) to calculating the roots of $q^5_0$ which sum to

$$\rho_0 = \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\,\right)} + \sqrt[3]{31\left(2-3\sqrt{-3}\,\right)} - 1}{3},$$

[38] The chosen generating square here is $4 \in \mathbb{F}_{29}$. It might be replaced by its sqaure 16.

we get the values

$$\rho_k^5 = \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\,\right)}\, z_+(\xi_5^k) + \sqrt[3]{31\left(2-3\sqrt{-3}\,\right)}\, z_-(\xi_5^k) - z_0(\xi_5^k)}{6},$$

where

$$z_0(x) := 280\,x^4 + 2140\,x^3 - 30\,x^2 - 960\,x - 1208,$$

$$z_\pm(x) := -50\,x^4 - 215\,x^3 - 60\,x^2 + 255\,x + 274 \pm \sqrt{-3}\left(60\,x^4 - 5\,x^3 + 130\,x^2 - 115\,x - 44\right),$$ [39]

and so we might proceed with constructing the heiskaitriacontagon via quintisection [4].[40]

## Instead of a conclusion

Instead of proceeding with infinitely more constructions of regular polygons via minimal angle section we would (temporarily) pause and return to that article which we mentioned first [6]. It is unexpectedly concluded with the following citation of Gauss, as translated by A. Clarke [5]:[41]

As a result the division of the whole circle into $n$ [a prime] parts requires, first, the division of the whole circle into $n-1$ parts; second, the division into $n-1$ parts of another arc which can be constructed as soon as the first division is accomplished; third, the extraction of one square root, and it can be shown that this is always $\sqrt{n}$.

We shall delegate to others the task of determining who and to what extent was misunderstood by whom before the latter erroneous conclusion was made (and carried on from one source to another). We would merely note that dividing the whole circle into $n$ [a prime] parts never requires the division of the whole circle or any other arc into $n-1$ parts. A division into (no more than) $(n-1)/2$ parts would always suffice. On the other hand, an extraction of a (single) square root (such as $\sqrt{n}$ ) would not suffice for constructing the arc to be divided into $(n-1)/2$ parts.[42] The construction of such arc would require the radicals necessary for solving a corresponding cyclotomic equation which roots would lie in an extension of $\mathbb{Q}$ of degree not necessarily less than $(n-1)/2-1$.

Gauss, perhaps, was reminding us that the cyclotomic polynomial $x^n-1$ is divisible by $x-1$ and that the quotient polynomial $(x^n-1)/(x-1)$, if $n$ is (an odd) prime, "always" splits over a quadratic extension of $\mathbb{Q}$ into a product of complex conjugate polynomial pair. That quadratic extension, as he determined, was obtained either by adjoining $\sqrt{n}$, if $n$ is congruent with $1 \pmod 4$, or by adjoining $\sqrt{-n}$, if $n$ is congruent with $3 \pmod 4$.[43]

## Acknowledgment

This article has been written to be sold (at an exponentially rising price) to Della Dumbaugh, provided that she guarantees its free accessibility to the readers of the American Mathematical Monthly worldwide. Meanwhile, as she shies away from the lime light, we tell her full (uncensored) story here.

[39] The chosen generating cube here is 29. It might be replaced with 2 which is a primitive fifth root of $1 \in \mathbb{F}_{31}$.

[40] Angle trisection is also required for constructing the 31-gon which was (therefore) not included between the 11-gon and the 41-gon in that last sentence in [6] (that was cited for us by Della Dumbaugh).

[41] It just preceded the last paragraph of [6] which last sentence was cited for us in that letter of Della Dumbaugh (which we mentioned in the introduction).

[42] Thus, contrary to the claim, a single square root extraction cannot be shown to "always" suffice (since it rarely does).

[43] And so, "technically speaking", an extraction of a single square root (of $n$) is required in both cases.

No commercial software companies, such as “Wolfram Alpha”, are permitted to use any presented construction or algorithm without an explicit written permission by the author.

# References


[1] S. Adlaj. A hendecagon construction via quintisection. Available at https://SemjonAdlaj.com/Excerpts/Hendecagon.pdf

[2] S. Adlaj. A 23-gon construction via 5 and 11 angle section. Available at https://SemjonAdlaj.com/Excerpts/Icositrigon.pdf

[3] S. Adlaj. A 29-gon construction via 3 and 7 angle section. Available at https://SemjonAdlaj.com/Excerpts/Icosienneagon.pdf

[4] S. Adlaj. A heiskaitriacontagon construction via quintisection. Available at https://SemjonAdlaj.com/Excerpts/Heiskaitriakontagon.pdf

[5] C. F. Gauss. Disquisitiones Arithmeticae. Yale University Press edition of 1966, New Haven and London. Translation from Latin language edition of 1870: Disquisitiones Arithmeticae edited by E. C. J. Schering. Translated by Arthur A. Clarke. Revised by William C. Waterhouse with the help of Cornelius Greither and A. W. Grootendorst. Available at https://doi.org/10.1007/978-1-4939-7560-0.

[6] A. Gleason. Angle Trisection, the Heptagona, and the Triskaidecagon. The American Mathematical Monthly, Vol. 95, No. 3 (March, 1988): 185-194. Available at https://www.jstor.org/stable/2323624

[7] D.H. Lehmer. A Note on Trigonometric Algebraic Numbers. The American Mathematical Monthly, Vol. 40, No. 3 (March, 1933): 165-166. Available at https://www.jstor.org/stable/2301023

[8] W. Watkins & J. Zeitlin. The Minimal Polynomial of $\cos(2\pi/n)$. The American Mathematical Monthly, Vol. 100, No. 5 (May, 1993): 471-474. Available at https://www.jstor.org/stable/2324301