A few regular polygons and Della Dumbaugh

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Carl Gauss and Della Dumbaugh

Carl Gauss was the "hero" of A. Gleason's paper "Angle Trisection, the Heptagona, and the Triskaidecagon" [5],[1] whereas the "hero" of our paper "A few regular polygons and Della Dumbaugh" is Della Dumbaugh. Unlike Gauss she has constructed nothing but unfortunately for the American Mathematical Monthly she became its editor and, unfortunately for her, she was misled to believe that she was either a Mathematician or a historian of Mathematics, whereas she is neither.[2] A historian of Mathematics (but not Della Dumbaugh), although need not necessarily personally contribute to Mathematics, would, of course, distinguish an explicit construction of a heptagon and a triskaidecagon, via trisection, from a mere statement concerning the possibility of constructing the hendecagon, via quintisection. The latter statement, made in [5], was not followed by an "offer for the reader to carry on the details" of constructing an hendecagon, via quintisection, unlike the construction of an enneakaidecagon,[3] via "two trisections", which "details were left to the reader!"[4] Yet, Della Dumbaugh told us in a letter that there is "little in this" (hendecagon construction via quintisection) "that has not been known for 35+ years."[5]

Calculating the minimal polynomial of the algebraic integer $2\cos\left(\frac{2\pi}{2n+1}\right)$

Let S^n denote the set of n natural numbers from 1 to n and let S^n_k denote its k-subset, that is, a subset with k many elements in it. The coefficients of a given monic polynomial p, which degree is n, are expressible, via Vieta's formulas, as symmetric functions of its n roots r_i, $1 \leq i \leq n$:

$$p(x) = \prod_{i \in S^n} (x - r_i) = \sum_{k=0}^{n} c_k \, x^k, \;\; c_k = (-1)^{n-k} \sigma^{n-k}, \;\; \sigma^k = \sum_{S^n_k \subseteq S^n} \prod_{i \in S^n_k} r_i.$$

Note that the sum in Vieta's formula for the "k^{th} elementary symmetric polynomial" σ^k contains $n!/k!/(n-k)!$ terms since there are that many subsets S^n_k of S^n, so the formula extends to calculating the "zeroth elementary symmetric polynomial" σ_0 as the sum containing a single term which is the empty product. Thus, $\sigma^0 = 1$ and $c_n = (-1)^0 \sigma^0 = 1$, as it ought to be.

[1] That paper was published in 1988 by the American Mathematical Monthly which editor then was Herbert Saul Wilf (June 13, 1931 – January 7, 2012).

[2] Actually, Della Dumbaugh is a podcaster who has unveiled herself in her own words which were (unfortunately) published

[3] The names for the polygons are based on combining a Greek-derived numerical prefix with the suffix "gon".

[4] The heptatriacontagon would be the "next" polygon to be constructed via trisections.

[5] Apparently, she "knew it" since she went to college and left us pondering on what was the "little in this" that has not been known to her.

For every positive integer n denote with p^n the monic polynomial, of degree n, which roots are

$$r_k^n = 2\cos\left(\frac{2\pi k}{2n+1}\right) = e^{2\pi k\sqrt{-1}/(2n+1)} + e^{-2\pi k\sqrt{-1}/(2n+1)},\ 1 \le k \le n,$$

and let c_k^n denote its coefficients, that is,

$$p^n(x) = \sum_{k=0}^{n} c_k^n\, x^k.$$

The coefficients of p^{n+1} might efficiently be calculated via an induction (on the degree n) as

$$c_k^{n+1} = (-1)^{n-k} c_k^n + \frac{1-(-1)^{n-k}}{2}\, c_{k-1}^n = \begin{cases} c_k^n & \text{if } n-k \text{ is even,} \\ c_{k-1}^n - c_k^n & \text{if } n-k \text{ is odd.} \end{cases} \tag{1}$$

Extending the range of "possible" values of n to include zero as a "base value", we set an "induction base" as

$$c_k^0 = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \tag{2}$$

In particular, all coefficients c_k^n (for all natural numbers n and all integers k) are integers,[6] and so r_1^n is an algebraic integer for every natural numbers n.[7]

Here is a list of the polynomials p^n for $1 \le n \le 13$:[8]

$$p^1(x) = x + 1,$$

$$p^2(x) = x^2 + x - 1,$$
$$p^3(x) = x^3 + x^2 - 2x - 1,$$
$$p^4(x) = x^4 + x^3 - 3x^2 - 2x + 1,$$
$$p^5(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1,$$
$$p^6(x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1,$$
$$p^7(x) = x^7 + x^6 - 6x^5 - 5x^4 + 10x^3 + 6x^2 - 4x - 1,$$
$$p^8(x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1,$$
$$p^9(x) = x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1,$$
$$p^{10}(x) = x^{10} + x^9 - 9x^8 - 8x^7 + 28x^6 + 21x^5 - 35x^4 - 20x^3 + 15x^2 + 5x - 1,$$
$$p^{11}(x) = x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1,$$
$$p^{12}(x) = x^{12} + x^{11} - 11x^{10} - 10x^9 + 45x^8 + 36x^7 - 84x^6 - 56x^5 + 70x^4 + 35x^3 - 21x^2 - 6x + 1,$$

[6] Including the vanishing coefficients c_k^n, where either $k > n$ or $k < 0$.

[7] The minimal polynomial of r_1^n over $\mathbb{Q}$ need not, of course, coincide with p^n but it must evidently be a (monic) polynomial divisor of p^n with integer coefficients. In fact, we shall soon see that each polynomial p^n factors into a product of minimal polynomials which n roots in total are the algebraic integers r_k^n, $1 \le k \le n$.

[8] The coefficients of these polynomials, along with the coefficient c_0^0, listed in an ascending order of the exponents, give rise to a sequence yet to be included in the "On-Line Encyclopedia of Integer Sequences (OEIS)": {**1**, **1**, **1**, -1, **1**, **1**, -1, -2, **1**, **1**, 1, -2, -3, **1**, **1**, 1, 3, -3, -4, **1**, **1**, -1, 3, 6, -4, -5, **1**, **1**, -1, -4, 6, 10, -5, -6, **1**, **1**, 1, -4, -10, 10, 15, -6, -7, **1**, **1**, 1, 5, -10, -20, 15, 21, -7, -8, **1**, **1**, -1, 5, 15, -20, -35, 21, 28, -8, -9, **1**, **1**, -1, -6, 15, 35, -35, -56, 28, 36, -9, -10, **1**, **1**, 1, -6, -21, 35, 70, -56, -84, 36, 45, -10, -11, **1**, **1**, 1, 7, -21, -56, 70, 126, -84, -120, 45, 55, -11, -12, **1**, **1** ...}.

$$p^{13}(x) = x^{13}+x^{12}-12x^{11}-11x^{10}+55x^9+45x^8-120x^7-84x^6+126x^5+70x^4-56x^3-21x^2+7x+1.$$

We might, as well, write down an explicit formula

$$p^n(x) = \sum_{k=0}^{n} c_k^n\, x^k,\ c_k^n = (-1)^m \frac{(m+k)!}{k!\, m!}, \tag{3}$$

$$m = \frac{n-k}{2} + \frac{(-1)^{n-k}-1}{4} = \begin{cases} (n-k)/2 & \text{if } n-k \text{ is even,} \\ (n-1-k)/2 & \text{if } n-k \text{ is odd,} \end{cases}$$

according to which the domain for the coefficients c_k^n might be extended to include all integers k with the linear recurrence relation (1) still satisfied on the extended domain. We shall note, however, that the coefficients c_k^n, which did and still do vanish whenever $k > n$, would now, according to their formula (3), generate, for each integer n, a sequence $S_n = \{c_{-1}^n,\, c_{-2}^n,\, c_{-3}^n \dots\}$.[9]

Note that p^n is divisible by p^m if and only if $2n+1$ is divisible by $2m+1$ and so p^n is the minimal polynomial of r_1^n, over $\mathbb{Q}$, if and only if $2n+1$ is (an odd) prime. Denoting with p_n the minimal (monic) polynomial of r_1^n, over $\mathbb{Q}$, we must have

$$p^n(x) = \prod_{(2m+1)|(2n+1)} p_m(x) \tag{4}$$

upon further assigning the (exceptional) identity (for the case of $n = 0$) $p_0(x) = p^0(x) \equiv 1$.

We might thus proceed to factor each polynomial p^n into a product of minimal polynomials, over $\mathbb{Q}$, for $1 \le n \le 13$:

$$p^1(x) = p_1(x) = x+1,$$
$$p^2(x) = p_2(x) = x^2+x-1,$$
$$p^3(x) = p_3(x) = x^3+x^2-2x-1,$$
$$p^4(x) = p_1(x)p_4(x) = (x+1)(x^3-3x+1),$$
$$p^5(x) = p_5(x) = x^5+x^4-4x^3-3x^2+3x+1,$$
$$p^6(x) = p_6(x) = x^6+x^5-5x^4-4x^3+6x^2+3x-1,$$
$$p^7(x) = p_1(x)p_2(x)p_7(x) = (x+1)(x^2+x-1)(x^4-x^3-4x^2+4x+1),$$
$$p^8(x) = p_8(x) = x^8+x^7-7x^6-6x^5+15x^4+10x^3-10x^2-4x+1,$$
$$p^9(x) = p_9(x) = x^9+x^8-8x^7-7x^6+21x^5+15x^4-20x^3-10x^2+5x+1,$$
$$p^{10}(x) = p_1(x)p_3(x)p_{10}(x) = (x+1)(x^3+x^2-2x-1)(x^6-x^5-6x^4+6x^3+8x^2-8x+1),$$

[9]Note that in the case of negative values of k, the assumption of the vanishing coefficients c_k^0, as given in (2), no longer agrees with formula (3) which, as we remarked earlier, would generate a family of sequences S_n. We might, in particular, enlist 13 terms of the sequence $S_0 = \{1,1,2,3,6,10,20,35,70,126,252,462,924\dots\}$. Remarkably, however, that recurrence relation (1), upon alternatively extending it to include S_0 as a "base sequence", would still be consistent with the identity $c_{-1}^n = 0$, for all $n > 0$, which we had earlier upon assuming (2). Moreover, the first n terms of each sequence S_n are guaranteed to vanish (without requiring the terms c_k^0 to vanish for negative k)! Let us demonstrate this observation with enlisting 13 terms of each of the following successive sequences: $S_1 = \{0,1,1,3,4,10,15,35,56,126,210,462,792\dots\}$, $S_2 = \{0,0,1,1,4,5,15,21,56,84,210,330,792\dots\}$, $S_3 = \{0,0,0,1,1,5,6,21,28,84,120,330,495\dots\}$, $S_4 = \{0,0,0,0,1,1,6,7,28,36,120,165,495\dots\}$, $S_5 = \{0,0,0,0,0,1,1,7,8,36,45,165,220\dots\}$, $S_6 = \{0,0,0,0,0,0,1,1,8,9,45,55,220\dots\}$, $S_7 = \{0,0,0,0,0,0,0,1,1,9,10,55,66\dots\}$, $S_8 = \{0,0,0,0,0,0,0,0,1,1,10,11,66\dots\}$, $S_9 = \{0,0,0,0,0,0,0,0,0,1,1,11,12\dots\}$, $S_{10} = \{0,0,0,0,0,0,0,0,0,0,1,1,12\dots\}$, $S_{11} = \{0,0,0,0,0,0,0,0,0,0,0,1,1\dots\}$.

$$p^{11}(x) = p_{11}(x) = x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1,$$
$$p^{12}(x) = p_2(x)p_{12}(x) = (x^2 + x - 1)(x^{10} - 10x^8 + 35x^6 + x^5 - 50x^4 - 5x^3 + 25x^2 + 5x - 1),$$
$$p^{13}(x) = p_1(x)p_4(x)p_{13}(x) = (x + 1)(x^3 - 3x + 1)(x^9 - 9x^7 + 27x^5 - 30x^3 + 9x + 1).$$

Denoting the degree of a polynomial p with $d(p)$, we might (recursively) deduce from expression (4), for p^n, a formula for the degree of the minimal polynomial p_n, that is,

$$d(p_n) = d(p^n) - \sum_{(2m+1)\in T^n} d(p_m) = n - \tau(n) = \pi(n),^{10}$$

where T^n denotes the set of proper divisors of $2n+1$, $\tau(n)$ is the number of elements of S^n which share a non-trivial divisor with $2n+1$ and $\pi(n)$ is the number of elements of S^n which are coprime with $2n+1$.[11] For $n \neq 0$, $\pi(n) = \phi(2n+1)/2$, where $\phi(\cdot)$ is the Euler's Totient function, which means that our formula agrees with the formula derived in [6]. Note, however, that $\phi(1) = 1$ which means that the agreement ceases in the exceptional case with $n = 0$.

Evidently, if $2n + 1$ is (an odd) prime then $p_n = p^n$ and (consequently) $d(p_n) = d(p^n) = n$.

Polynomial factorization towards the construction of regular polygons

The degree of any minimal polynomial p_n matches the degree of its splitting field, over $\mathbb{Q}$, since any field extension of $\mathbb{Q}$ containing a root of p_n necessarily contains all of its roots.[12] No intermediate field extensions between $\mathbb{Q}$ and the splitting field for p_n exist whenever both n and $2n + 1$ are primes.[13]

An intermediate factorization of a polynomials p_n, where $2n+1$ is composite, might be attained by adjoining to $\mathbb{Q}$ a root of p_m, where $2m + 1$ is a proper divisor of $2n + 1$. In particular,

$$p_7(x) = x^4 - x^3 - 4x^2 + 4x + 1 = \left(x^2 + r_1^2\, x + r_2^2 - 1\right)\left(x^2 + r_2^2\, x + r_1^2 - 1\right),$$

$$p_{10}(x) = x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 =$$
$$= \left(x^2 + r_1^3\, x + (r_1^3)^2 - 3\right)\left(x^2 + r_2^3\, x + (r_2^3)^2 - 3\right)\left(x^2 + r_3^3\, x + (r_3^3)^2 - 3\right),$$
$$p_{12} = x^{10} - 10x^8 + 35x^6 + x^5 - 50x^4 - 5x^3 + 25x^2 + 5x - 1 = (x^5 - 5\,x^3 + 5\,x - r_1^2)(x^5 - 5\,x^3 + 5\,x - r_2^2),$$
$$p_{13} = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x + 1 = \left(x^3 - 3\,x - r_1^4\right)\left(x^3 - 3\,x - r_2^4\right)\left(x^3 - 3\,x - r_3^4\right).$$

An intermediate factorization of a polynomials p_n, where $2n + 1$ is prime, is attainable as long as n is composite.

$$p_6(x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1 =$$
$$= \left(x^3 + (u_{13}^+ - 1)\, x^2 - x - u_{13}^+\right)\left(x^3 + (u_{13}^- - 1)\, x^2 - x - u_{13}^-\right),$$
$$p_8(x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 =$$

[10] Thus, the formulas for $d(p_n)$ give rise to yet another sequence, which upon choosing an offset 1, would coincide with the sequence A072451 in OEIS: $1, 2, 3, 3, 5, 6, 4, 8, 9, 6, 11, 10, 9 \ldots$.

[11] We must point out here that our chosen notation is reminiscent of the widespread notational convention (but does not coincide with it), where $\pi(n)$ is the number of primes in S^n. Note, as well, that the (non-prime) number 1 is coprime with all natural numbers. And since the number of elements of the empty set is 0 (and so must be the number of elements of the empty set which are coprime with 1) we must have $\pi(0) = 0$, being the degree of the constant polynomial 1.

[12] In other words, adjoining a root of p_n to $\mathbb{Q}$ is a Galois extension of $\mathbb{Q}$.

[13] The corresponding sequence is A079148 in OEIS which is the subsequence of primes p, where "$p - 1$ has at most 2 prime factors". These are the primes $3, 5, 7, 11, 23 \ldots$

$$= \left(x^4 + u_{17}^+ x^3 - (u_{17}^- + 1)\, x^2 + (2\, u_{17}^- + 1)\, x - 1\right)\left(x^4 + u_{17}^- x^3 - (u_{17}^+ + 1)\, x^2 + (2\, u_{17}^+ + 1)\, x - 1\right) =$$

$$\left(x^2 + \frac{u_{17}^- + v_{17}^-}{2}\, x - \frac{u_{17}^+ + v_{17}^+}{2}\right)\left(x^2 + \frac{u_{17}^- - v_{17}^-}{2}\, x - \frac{u_{17}^+ - v_{17}^+}{2}\right) \cdot$$

$$\cdot \left(x^2 + \frac{u_{17}^+ + v_{17}^+}{2}\, x - \frac{u_{17}^- - v_{17}^-}{2}\right)\left(x^2 + \frac{u_{17}^+ - v_{17}^+}{2}\, x - \frac{u_{17}^- + v_{17}^-}{2}\right),$$

$$p_9(x) = x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1 =$$

$$(x^3 + (\beta^3 + 1)x^2 + \beta^2 x + \beta^1)(x^3 + (\beta^1 + 1)x^2 + \beta^3 x + \beta^2)(x^3 + (\beta^2 + 1)x^2 + \beta^1 x + \beta^3),$$

where

$$u_{13}^{\pm} := \frac{3 \pm \sqrt{13}}{2},\ u_{17}^{\pm} := \frac{1 \pm \sqrt{17}}{2},\ v_{17}^{\pm} := \sqrt{\pm\sqrt{17}\, u_{17}^{\pm}}\ ,^{14}$$

$$\beta^s = -\frac{e^{2\pi s\sqrt{-1}/3}\gamma_- + e^{-2\pi s\sqrt{-1}/3}\gamma_+ + 2}{3},\ \gamma_{\pm} = \left(\frac{19\,(7 \pm 3\sqrt{-3}\,)}{2}\right)^{1/3}.$$

A construction of triskaidecagon (via angle trisection) based on calculating the roots of p_6 was offered in [5]. The same source indicates that Gauss constructed the 17-gon in 1796 and states (as we mentioned earlier) that the construction of the 19-gon requires two trisections which "details were left to the reader".

No intermediate factorization exists for of p_n if both $2n + 1$ and n are primes. Thus, for any given field extension of $\mathbb{Q}$, a polynomial such as p_5 or p_{11} either remains irreducible or (exclusively and completely) splits into linear factors. Constructing the roots of these solvable polynomials (p_5 and p_{11}) enable a construction of a hendecagon via quintisection and a construction of a 23-gon via 5 and 11 angle section, respectively.

Expressing in radicals the roots of the solvable polynomials p^5 and p^{11}

In order to solve the polynomial p^5 we introduce the Lagrange resolvent as

$$\rho(\eta) := r_1^5\, \eta + r_2^5\, \eta^2 + r_4^5\, \eta^3 + r_3^5\, \eta^4 + r_5^5\,,^{15}$$

where η is a fifth, not necessarily primitive, root of unity.

Evidently, the value of the Lagrange resolvent does depend on the chosen fifth root of unity. Subsequently, for the sake of brevity, we shall denote $\rho(\xi_5^k)$, where $\xi_5 := e^{2\,\pi\,\sqrt{-1}/5}$, by ρ_k. So, in particular, $\rho_0 = \rho(1) = -1$, that is, the sum of the roots of p^5.

Now note that

$$5\, r_5^5 = \rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4,$$

so the quintuples of the roots of p^5 are expressible as sums of fifth roots of the five values ρ_k^5 all of which might be calculated once we calculate

$$\rho_1^5 = 90\, \xi_5^4 - 20\, \xi_5^3 + 255\, \xi_5^2 - 130\, \xi_5 - 196$$

and proceed to replace ξ_5 (on the right-hand side) by the fifth roots of unity ξ_5^k,[16] including the trivial root for which $k = 0$, corresponding to the value $\rho_0^5 = -1$ (as it ought to be since we already

[14] Only the upper (exclusively) or the ower sign is meant to be taken on both sides of an equation.

[15] There is no typing error here, as might naively be perceived, that is, we ought not swap r_4^5 with r_3^5 without simultaneously swapping η^3 with η^4.

[16] For $k \in \{1, 2, 3, 4\}$, the set of the four values $\{k,\ 2k,\ 3k,\ 4k\}$ is the same (mod 5) as the set $\{1, 2, 3, 4\}$.

know that $\rho_0 = -1$). The four values, corresponding to four primitive fifth roots of unity, might be paired as two complex conjugate pairs $u_\pm$, $v_\pm$, where

$$\rho_1^5 = u_+,\ \rho_2^5 = v_+,\ \rho_3^5 = v_-,\ \rho_4^5 = u_-,$$

$$u_\pm = -\frac{11\left(89 + 25\sqrt{5} \pm 5\sqrt{178\sqrt{5} - 410}\right)}{4},\ v_\pm = -\frac{11\left(89 - 25\sqrt{5} \pm 5\sqrt{-178\sqrt{5} - 410}\right)}{4},$$

and so ρ_1^5, ρ_2^5, ρ_3^5 and ρ_4^5 are the four roots of the polynomial

$$\left(x^2 + \frac{11(89 + 25\sqrt{5})}{2}\,x + 11^5\right)\left(x^2 + \frac{11(89 - 25\sqrt{5})}{2}\,x + 11^5\right) =$$

$$= x^4 + 89 \cdot 11\,x^3 + 13 \cdot 33^3\,x^2 + 89 \cdot 11^6\,x + 11^{10}.$$

Formidable attempts for constructing a regular hendecagon were ignited at "the Mathematics Stack Exchange" on July 31, 2016 with "Constructing the 11-gon by splitting an angle in five" and further discussed in "Simplifying a radical-trigonometric expression for the hendecagon angle" till August 23, 2023. However, the construction on June 13, 2022 in "Radical representation of $\cos\frac{2\pi}{11}$" came even closer to the construction which we publicized on December 14, 2024.[17]

The preceding calculation of the roots of p^5 might be generalized to calculating the roots of p^n whenever both n and $2n+1$ are primes. We would then put

$$\rho_k := \rho(\xi_n^k),\ \rho(\eta) := \sum_{k=0}^{n-1} r_{2^k}^n \eta^k,\ \xi_n := e^{2\,\pi\,\sqrt{-1}/n},\ ^{18}$$

and observe that

$$n\,r_n^n = \sum_{k=0}^{n-1} \rho_k,\ ^{19}$$

so the n-multiples of the roots of p_n are expressible as sums of n^{th} roots of the $n-1$ values ρ_k^n, $1 \le k \le n-1$ (that is, $(n-1)/2$ complex conjugate pairs), in addition to the value $\rho_0 = -1$.[20]

In order to derive an explicit expression for ρ_k^n we shall denote by P_n the set of all integer partitions of n. To each partition $P_n^j \in P_n$ we might associate an n-tuple of integers $(j_1, \ldots, j_n)$, adhering to the conditions

$$\sum_{i=1}^{n} j_i = n,\ n \ge j_1 \ge \cdots \ge j_n \ge 0.$$

The alternating group A_{n-1} (which is the subgroup of even permutations of the corresponding symmetric group) acts on the (ordered) $n-1$-tuple $(j_2, \ldots, j_n)$ (via permuting the indices). So if we denote by Q_n^j the orbit of P_n^j under the said action of A_{n-1} and if $(k_1, \ldots, k_n) \in Q_n^j$ then $k_1 = j_1$, that is, j_1 remains "fixed in its position".[21]

[17] Only one last simplification was missing, thereby sending others on a wrong track of activating "Mathematica ToRadicals", leading to a horrible radical expression.

[18] When both n and $2n+1$ are primes, the multiplicative group (of order $2n$) of the field of $2n+1$ elements contains a maximal number $(n-1)$ of generators. The element 2, in particular, is either a generator or a square thereof, and so is guaranteed to generate no less than all n distinct squares, including the identity element. Thus and furthermore, 2 might be replaced by any of its powers 2^k, as long as k is not a multiple of n.

[19] For $k \in \{1, 2, 3, 4\}$, the set of the four values $\{k,\ 2k,\ 3k,\ 4k\}$ is congruent mod 5 with the set $\{1, 2, 3, 4\}$.

[20] All other n^{th} roots of $\rho_0^n = -1$ are excluded.

[21] Recall that the number of elements of the orbit Q_n^j might be obtained by dividing the order of the group A_{n-1} by the order of its subgroup which stabilizes $(j_2, \ldots, j_n)$.

Let s_k denote a sum of of k (strictly) positive terms and consider the "total" sum of n (not necessarily distinct) powers of 2

$$s_n = s_n(k_1, \cdots, k_n) := \sum_{i=1}^{n} k_i 2^i$$

for which we might construct a monomial

$$x^s,\ s = s_n - 2s_k \pmod{2n+1}, \tag{5}$$

corresponding to a "partial" sum s_k of (exactly) k summands (powers of 2). There are 2^{n-1} such (not necessarily distinct) monomials as k ranges from 0 to $(n-1)/2$, inclusively, and so we might define the polynomial $b(x) = b(x, k_1, \cdots, k_n)$ as the sum of these monomials.[22]

We are now ready to write down the sought for expression as

$$\rho_k^n = \sum_{P_n^j \in P_n} \frac{n!}{\alpha(j_1)} \prod_{i=1}^{n} \frac{1}{j_i!} \sum_{\{k_1,\ldots,k_n\} \in Q_n^j} \left((2n+1)b(0) - 2^{n-1}\right) \xi_n^{km},\ m := \sum_{i=1}^{n-1} k_i i \pmod{n},$$

where $\alpha(j_1)$ denotes the number of occurrences of j_1 in $(j_1, \ldots, j_n)$.[23]

The latter expression, with $n = 11$, was used in [2] for constructing an icositrigon for which

$$\rho_1^{11} = -5332019\,\xi_{11}^{10} - 1781164\,\xi_{11}^{9} - 2099438\,\xi_{11}^{8} - 5853705\,\xi_{11}^{7} + 290906\,\xi_{11}^{6} - 4039948\,\xi_{11}^{5} +$$

$$4283752\,\xi_{11}^{4} + 15151367\,\xi_{11}^{3} + 12996313\,\xi_{11}^{2} - 18426793\,\xi_{11} + 4810728.$$

It was not used, with $n = 3$, for constructing a heptagon for which

$$\rho_1^3 = -12\,\xi_3^2 + 9\,\xi_3 + 2 = \frac{7(1 + 3\sqrt{-3}\,)}{2}.$$[24]

The Legendre symbol of a given order and its applications

Let $\mathbb{F}_p$ denote a prime field of order p and let ξ_k denote a primitive k^{th} root of unity, where k is a divisor of $p-1$. Introduce the Legendre symbol of order k, which we shall denote as L_k^p, as a function on $\mathbb{F}_p$ which range includes zero along with the powers of ξ_k. The Legendre symbol of order k would then map zero to zero and is determined by a homomorphism mapping a generator g of the multiplicative group of $\mathbb{F}_p$ to ξ_k which generates a group isomorphic with the additive group $\mathbb{Z}_k$.[25] Explicitly, we have

$$L_k^p(0) = 0,\ L_k^p(g^l) = \xi_k^l.$$

Evidently, the Legendre symbol of order k depends on the choice of the generator g, unless k is either one or two. In fact, the widely accepted definition of the Legendre symbol (which order is not usually specified) agrees with our definition of the Legendre symbol of order 2.

[22] Note that $b(1) = 2^{n-1}$ (by construction).

[23] That number is no less than one and would evidently coincide with the number of occurrences of j_1 in $(j_1 = k_1, \ldots, k_n)$.

[24] The case of the heptagon is exceptional since A_2 is the trivial group and its action must be "augmented" with the action of the "full" symmetric group (with two elements).

[25] Note that we did not require k to be a prime.

We note that the value $b(0)$ which appeared in the preceding section might be expressed via a sum of Legendre symbols of order 1, that is,

$$b(0) = \sum_s 1 - L_1^{2n+1}(s),$$

where s ranges over all the sums defined in (5),[26] and so

$$(2n+1)b(0) - 2^{n-1} = 2n\lambda_1,$$

$$\lambda_1 = \lambda_1(k_1, \ldots, k_n) := \sum_s \left(1 - L_1^{2n+1}(s) - \frac{L_1^{2n+1}(s)}{2n}\right).$$

The formula for calculating the n^{th} power of the Lagrange resolvent when both n and $2n+1$ are primes might be extended to instances when both n and $4n+1$ are primes and to instances when both n and $6n+1$ are primes.[27] The minimal polynomial over $\mathbb{Q}$ would respectively be p^n, p^{2n} and p^{3n}. The polynomial p^{2n} (p^{3n}) might then be factored over a quadratic (cubic) extension of $\mathbb{Q}$ into a product of two (three) polynomials of degree n, so we have

$$p^{2n} = q_+^n(x)\, q_-^n(x),\ p^{3n} = q_0^n(x)\, q_1^n(x)\, q_2^n(x),$$

where we might further assume that the roots of $q_\pm^n$ are those r_j^{2n} for which $L_2^{4n+1}(j) = \pm 1$, whereas the roots of q_i^n are those r_j^{3n} for which $L_3^{6n+1}(j) = \xi_3^i$.[28] The sum of the roots of $q_\pm^n$ is $(\pm\sqrt{4n+1} - 1)/2$, whereas the sum of the roots of q_i^n is $(\sqrt{6n+1}\,(\xi_3^i u + \xi_3^{-i} u^{-1}) - 1)/3$, where u is a cube root of unit modulus.[29]

We would then have

$$\rho_k^n = \sum_{P_n^j \in P_n} \frac{2n}{\alpha(j_1)} \prod_{i=1}^{n} \frac{i}{j_i!} \sum_{\{k_1,\ldots,k_n\} \in Q_n^j} \lambda\,(k_1, \ldots, k_n)\ \xi_n^{km}, \tag{6}$$

$$m := \sum_{i=1}^{n-1} k_i i \pmod n,\ \lambda\,(k_1, \ldots, k_n) =$$

$$\begin{cases} \lambda_1 = \sum_s \left(1 - L_1^{2n+1}(s) - \frac{L_1^{2n+1}(s)}{2n}\right) & \text{if } n \text{ and } 2n+1 \text{ are primes,} \\ \lambda_2 = \sum_s \left(1 - L_1^{4n+1}(s) - \frac{L_1^{4n+1}(s) - \sqrt{4n+1}\, L_2^{4n+1}(s)}{4n}\right) & \text{if } n \text{ and } 4n+1 \text{ are primes,} \\ \lambda_3 = \sum_s \left(1 - L_1^{6n+1}(s) - \frac{L_1^{6n+1}(s) - \sqrt{6n+1}\left(u\, L_3^{6n+1}(s) + (u\,(L_3^{6n+1}(s))^{-1}\right)}{6n}\right) & \text{if } n \text{ and } 6n+1 \text{ are primes,} \end{cases}$$

[26] Recall that there are 2^{n-1} such sums so the values s might be repeated.

[27] These are, of course, not mutually exclusive instances. For example, $13 = 4 \cdot 3 + 1 = 6 \cdot 2 + 1$. Amusingly, for a prime $n = 3$, we have the primes $2n+1 = 7$, $2 \cdot 2n + 1 = 13$, $2 \cdot 3n + 1 = 19$, $2 \cdot 5n + 1 = 31$, $2 \cdot 7n + 1 = 43$, $2 \cdot 11n + 1 = 67$, $2 \cdot 13n + 1 = 79$ and $2 \cdot 17n + 1 = 103$.

[28] Note that -1 is a square in $\mathbb{F}_{4n+1}$ and is a cube in $\mathbb{F}_{6n+1}$

[29] The two sums $(\pm\sqrt{4n+1} - 1)/2$ are the two roots of the quadratic polynomial $f_n(x) := x^2 + x - n$. The three sums $(\sqrt{6n+1}\,(\xi_3^i u + \xi_3^{-i} u^{-1}) - 1)/3$ are, in turn, the roots of the cubic polynomial

$$g_n(x) := x^3 + x^2 - 2n\,x - \frac{(6n+1)\,p_n - 4n^2}{3},$$

where p_n is the number of partitions of $1 \in \mathbb{F}_{6n+1}$ into a sum of two non cubes which product is a cube. Note that $p_1 = 1$ (since 3+5 is the only partition of $1 \in \mathbb{F}_7$ meeting the requirements) and so $g_1(x) = x^3 + x^2 - 2\,x - 1 = p^3(x)$. A short list of a few more values would be $p_2 = 1$, $p_3 = 3$, $p_5 = 4$, $p_7 = 4$, $p_{11} = 7$, $p_{13} = 7$. It corresponds to the cubic polynomials $g_2(x) = x^3 + x^2 - 4\,x + 1$, $g_3(x) = x^3 + x^2 - 6\,x - 7$, $g_5(x) = x^3 + x^2 - 8\,x - 10$, $g_7(x) = x^3 + x^2 - 14\,x + 8$, $g_{11}(x) = x^3 + x^2 - 22\,x + 5$, $g_{13}(x) = x^3 + x^2 - 26\,x + 41$.

where $s = s_n - 2\,s_k$ is an element from the domain of the corresponding Legendre symbol, so it must be regarded accordingly as an element from $\mathbb{F}_{2n+1}$, $\mathbb{F}_{4n+1}$ or $\mathbb{F}_{6n+1}$, whereas

$$s_n = s_n(k_1, \cdots, k_n) := \sum_{i=1}^{n} k_i\, g^i,$$

with g accordingly is either a generator,[30] a square of a generator or a cube of generator of the corresponding multiplicative group of either $\mathbb{F}_{2n+1}$, $\mathbb{F}_{4n+1}$ or $\mathbb{F}_{6n+1}$.

The case of the prime $13 = 6 \cdot 2 + 1 = 4 \cdot 3 + 1$ must be dealt with separately.[31] So we might write

$$2\, r_1^6 = \rho_0 + \rho_1,$$[32]

where

$$\rho_0 = r_1^6 + r_5^6 = \frac{\sqrt{13}}{3}\left(\sqrt[3]{\frac{-5+3\,\sqrt{-3}}{26}} + \sqrt[3]{\frac{-5-3\,\sqrt{-3}}{26}}\right) - \frac{1}{3},$$[33]

and ρ_1^2 is the discriminant of the quadratic polynomial $x^2 - \rho_0\, x + 2\,\lambda_3(1,1)$ which we might explicitly calculate as

$$\rho_1^2 = 4\,(\lambda_3\,(2,0) - \lambda_3\,(1,1)) = \frac{13}{3} + \frac{\sqrt{13}}{3}\left(\sqrt[3]{\frac{131+15\,\sqrt{-3}}{26}} + \sqrt[3]{\frac{131-15\sqrt{-3}}{26}}\right).$$

Or we might, alternatively, write

$$3\, r_1^6 = \rho_0 + \rho_1 + \rho_2$$

where

$$\rho_0 = r_1^6 + r_3^6 + r_4^6 = \frac{\sqrt{13}-1}{2}$$

and the complex conjugate pair $13 + \sqrt{13}\,(3\,\xi_3^{\pm} - 1)$ must coincide with the pair ρ_1^3, ρ_2^3.

We shall not avoid the case of the prime $19 = 6 \cdot 3 + 1$ (which has awaited us "for 35+ years"), where we already know that

$$\rho_0 = r_8^9 + r_7^9 + r_1^9 = \frac{\sqrt[3]{(7+3\sqrt{-3}\,)\,19/2} + \sqrt[3]{(7-3\sqrt{-3}\,)19/2} - 1}{3},$$[34]

so we calculate the complex conjugate pair ρ_1^3, ρ_2^3 as

$$\left(r_8^9\,\xi_3^{\pm 1} + r_7^9\,\xi_3^{\mp 1} + r_1^9\right)^3 = \frac{38 + \sqrt[3]{(11 \mp 21\sqrt{-3}\,)\,38^2} + \sqrt[3]{(3751 \mp 7959\sqrt{-3}\,)19/2}}{3}.$$[35]

[30] A generator might be replaced by its square but not vice versa, that is, a generator ought not replace its square.

[31] Only one of the two one-element subsets of a two-element set would be taken in order to calculate a sum s as $s_2 - 2s_1 \in \mathbb{F}_{13}$. That value $L_3^{13}(s)$, however, does not depend on the chosen value of s_1 since -1 is a cube in $\mathbb{F}_{13}$.

[32] Tacitly assuming the positivity of ρ_1 and so we must (consequently) have $\rho_0 - \rho_1 = r_5^6$.

[33] The cube roots here lie in the first and fourth quadrants, respectively.

[34] This equality is better viewed as a definition of the right-hand side, where particular complex conjugate branches of the cube root (from the fourth and the first quadrant, respectively) must be taken, via the left.

[35] The value of a cube root here lies in the quadrant of its argument.

Expressing in radicals the roots of the solvable polynomials p^{14} and p^{15}

The minimal polynomial p^{14} factors over a quadratic extension of $\mathbb{Q}$ into a product of two septic polynomials, that is,

$$p^{14}(x) = q_+^7(x)\, q_-^7(x),\ q_\pm^7(x) = x^7 - \frac{\pm\sqrt{29}-1}{2}\, x^6 - 3\, x^5 + \frac{\pm 5\sqrt{29}-9}{2}\, x^4 +$$

$$+\frac{\pm\sqrt{29}-11}{2}\, x^3 - \frac{\pm 5\sqrt{29}-19}{2}\, x^2 - (\pm 2\sqrt{29}-13)\, x - \frac{\pm\sqrt{29}-5}{2},$$

whereas the minimal polynomial p^{15} factors over a cubic extension of $\mathbb{Q}$ into a product of three quintic polynomials, that is,

$$p_{15}(x) = q_0^5(x)\, q_1^5(x)\, q_2^5(x),\ q_i^5(x) = x^5 - \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\right)}\,\xi_3^i + \sqrt[3]{31\left(2-3\sqrt{-3}\right)}\,\xi_3^{-i} - 1}{3}\, x^4 +$$

$$+\frac{\sqrt[3]{31\left(2+3\sqrt{-3}\right)}\,\xi_3^{i-1} + \sqrt[3]{31\left(2-3\sqrt{-3}\right)}\,\xi_3^{1-i} - 4}{3} x^3 +$$

$$+\frac{\sqrt[3]{31\left(101+12\sqrt{-3}\right)}\,\xi_3^{i} + \sqrt[3]{31\left(101-12\sqrt{-3}\right)}\,\xi_3^{-i} + 8}{3}\, x^2 +$$

$$+\frac{2\left(\sqrt[3]{31\left(2+3\sqrt{-3}\right)}\,\xi_3^{i-2} + \sqrt[3]{31\left(2-3\sqrt{-3}\right)}\,\xi_2^{2-i} - 4\right)}{3}\, x - 1.$$

Applying (6) to calculating the roots of q_+^7 which sum to

$$\rho_0 = \frac{\sqrt{29}-1}{2},$$

we get

$$\rho_1^7 = 10885\,\xi_7^6 + 3171\,\xi_7^5 - 11039\,\xi_7^4 - \frac{40145}{2}\,\xi_7^3 + \frac{5327}{2}\,\xi_7^2 + \frac{63385}{2}\,\xi_7 - 18869 +$$

$$\sqrt{29}\,(-3605\,\xi_7^6 + 5761\,\xi_7^5 + 840\,\xi_7^4 - \frac{4137}{2}\,\xi_7^3 + \frac{6601}{2}\,\xi_7^2 - \frac{3563}{2}\,\xi_7 - 2110).^{36}$$

The latter expression might be restricted to the three vertices of the heptagon ξ_7, ξ_7^2, ξ_7^4:

$$\rho_1^7 = \frac{7(18879 - 1709\sqrt{29} - 6641\sqrt{-7} - 2237\sqrt{-203}\,)}{4}\,\xi_7^2 +$$

$$+\frac{7(19981 - 6411\sqrt{29} - 8845\sqrt{-7} + 439\sqrt{-203}\,)}{4}\,\xi_7 - \frac{84071}{2} - 8557\sqrt{29} - 10962\sqrt{-7} + \frac{4445\sqrt{-203}}{2},$$

which would, upon cyclically permuting the (ordered) three verices, generate the three values ρ_1^7, ρ_2^7, ρ_4^7, thereby enabling the construction of the icosienneagon via 3 and 7 angle section.

Applying (6) to calculating the roots of q_0^5 which sum to

$$\rho_0 = \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\right)} + \sqrt[3]{31\left(2-3\sqrt{-3}\right)} - 1}{3},$$

[36] The chosen generating square here is $4 \in \mathbb{F}_{29}$.

we get the values

$$\rho_k^5 = \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\,\right)}\, z_+(\xi_5^k) + \sqrt[3]{31\left(2-3\sqrt{-3}\,\right)}\, z_-(\xi_5^k) - z_0(\xi_5^k)}{6},$$

where

$$z_0(x) := 280\,x^4 + 2140\,x^3 - 30\,x^2 - 960\,x - 1208,$$

$$z_\pm(x) := -50\,x^4 - 215\,x^3 - 60\,x^2 + 255\,x + 274 \pm \sqrt{-3}\left(60\,x^4 - 5\,x^3 + 130\,x^2 - 115\,x - 44\right),$$ [37]

and so we might proceed with constructing the heiskaitriacontagon via quintisection.

Instead of a conclusion

Instead of proceeding with infinitely more constructions of regular polygons via minimal angle section we would (temporarily) pause and return to that article which we mentioned first [5]. It is unexpectedly concluded with the following citation of Gauss, as translated by A. Clarke [8]:[38]

As a result the division of the whole circle into n [a prime] parts requires, first, the division of the whole circle into $n-1$ parts; second, the division into $n-1$ parts of another arc which can be constructed as soon as the first division is accomplished; third, the extraction of one square root, and it can be shown that this is always $\sqrt{n}$.

We shall delegate to others the task of determining who and to what extent was misunderstood by whom before the latter erroneous conclusion was made (and carried on from one source to another). We would merely note that dividing the whole circle into n [a prime] parts never requires the division of the whole circle or any other arc into $n-1$ parts. A division into (no more than) $(n-1)/2$ parts would always suffice. On the other hand, an extraction of a (single) square root (such as $\sqrt{n}$) would not suffice for constructing the arc to be divided into $(n-1)/2$ parts.[39] The construction of such arc would require the radicals necessary for solving a corresponding cyclotomic equation which roots would lie in an extension of $\mathbb{Q}$ of degree not necessarily less than $(n-1)/2-1$.

Gauss, perhaps, was reminding us that the cyclotomic polynomial x^n-1 is divisible by $x-1$ and that the quotient polynomial $(x^n-1)/(x-1)$, if n is (an odd) prime, splits over a quadratic extension of $\mathbb{Q}$ into a product of complex conjugate polynomial pair. That quadratic extension, as he determined, was obtained either by adjoining $\sqrt{n}$, if n is congruent with 1(mod 4), or by adjoining $\sqrt{-n}$, if n is congruent with 3(mod 4).[40]

Acknowledgment

This article has been written to be sold (at an exponentially rising price) to Della Dumbaugh, provided that she guarantees its free accessibility to the readers of the American Mathematical Monthly worldwide. Meanwhile, as she weights her options, the readers who are curious to know her full (uncensored) story might access here.

[37] The chosen generating cube here is 29. It might be replaced with 2 which is a primitive fifth root of $1 \in \mathbb{F}_{31}$.

[38] It just preceded the last paragraph of [5] which last sentence was cited for us in that letter of Della Dumbaugh (which we mentioned in the introduction).

[39] Thus, contrary to the claim, a single square root extraction cannot be shown to "always" suffice (since it rarely does).

[40] Technically speaking, an extraction of a single square root (of n) is required in both cases.

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