Duality of base-coordinate transformation

The same transformation matrix T might be employed to specify a transition from an old (ordered) basis $\{\mathbf{e}_i\}_{i=1}^n$ to a new (ordered) basis $\{\mathbf{f}_i\}_{i=1}^n$, as well as, the converse transformation from the new coordinates $(y^i)_{i=1}^n$ to old coordinates $(x^i)_{i=1}^n$.

Assuming $x^i\mathbf{e}_i = y^i\mathbf{f}_i$, this duality is best expressed via two successive equalities:

$$(\mathbf{e}_1,\ \mathbf{e}_2, \cdots, \mathbf{e}_n)\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x_n \end{pmatrix} = (\mathbf{e}_1,\ \mathbf{e}_2, \cdots, \mathbf{e}_n)\ T\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix} = (\mathbf{f}_1,\ \mathbf{f}_2, \cdots, \mathbf{f}_n)\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix},$$

which might be read forward (from left to right), as well as, backward (from right to left). The choice merely depends upon choosing a first defining equality and confirming its reconcilability with the subsequent transformation, emerging from applying the second.