The Galois top invariant of motion

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The Galois top is a heavy top (in a uniform gravitational field) which fixed point O lies at the Galois axis. The notion of the Galois top was introduced at the PCA annual conference on April 17, 2024.

The coordinates of the centre of mass G of the Galois top with respect to a coordinate system which (directed) axes coincide with the principal axes of inertia, passing through the fixed point O, are

$$\left(\sqrt{\frac{C(A-B)}{B(A-C)}},\, 0,\, \sqrt{\frac{A(B-C)}{B(A-C)}}\right) d,$$

where d is the distance between the fixed point O and the centre of mass G, whereas A, B and C are the (ascendingly ordered) principal momenta of inertia about the fixed point O.

Along with the well-known "classical" invariants of motion,[1] which are the (total) energy and the "vertical" projection of the angular momentum, the Galois top possesses its "own" invariant of motion, which is

$$e^{\gamma\alpha\int_0^t q(t)\,dt}\big(\gamma\, r(t)+\alpha\, p(t)\big)\equiv\gamma\, r(0)+\alpha\, p(0),\ \gamma:=\sqrt{\frac{C-B}{A}},\ \alpha:=\sqrt{\frac{B-A}{C}},^{2}$$

where p, q and r are the respective projections of the angular velocity on the (three) principal axes of inertia,[3] corresponding to the (principal) momenta $A\le B\le C$.

The Lagrange top arises when either $A=B$ or $B=C$ and, respectively, either r or p is a constant function (of time).

The Euler top arises when the fixed point O coincides with the centre of mass G, where the Galois axes intersect and the afore-indicated motion invariant splits into two:

$$e^{\pm\gamma\alpha\int_0^t q(t)\,dt}g^{\pm}(t)\equiv g^{\pm}(0),\ g^{\pm}(t):=\gamma\, r(t)\pm\alpha\, p(t),$$

which product is a homogeneous (in r and p) binomial of degree 2, that is,

$$g^{+}(t)g^{-}(t)\equiv g^{+}(0)g^{-}(0)=\big(\gamma\, r(0)\big)^2-\big(\alpha\, p(0)\big)^2=\frac{m^2-Bh}{CA},$$

where $h:=Ap^2+Bq^2+Cr^2$ is twice the (conserved) energy and m is the (consequently constant) modulus of the angular momentum, that is, $m^2:=A^2p^2+B^2q^2+C^2r^2$.

For the Dzhanibekov top, in particular, the projection of the angular momentum on one (and only one) of the Galois axes (identically) vanishes,[4] that is, $g^{+}(t)g^{-}(t)\equiv 0$, whereas the projection on the "other" Galois axis "never" vanishes![5]

[1] We use the term "invariant" of motion to indicate a constant function (of time), although an archaic term "first integral" is still abundantly used.

[2] We are tacitly assuming that γ and α, being defined as square roots, share the same (positive) sign, corresponding to a particular choice of the Galois axis. Choosing oposite signs would correspond to choosing the "other" Galois axis.

[3] That is, p, q and r are the time-dependent coordinate-functions of the angular velocity.

[4] An intuitive description by E.A. Mityushov (in a letter dated February 27, 2018) of the Galois axis (for the Dzhanibekov top) upon which the projection of the angular momentum vanishes was cited (on page 6) in Dzhanibekov screw.

[5] Assuming the complex plane to be the time domain. The latter assertion does not hold if the complex plane is compactified (to the Riemann sphere) via augmenting the time domain with the (single) point of complex infinity, as discussed at the Russian Interdisciplinary Temporology Seminar on November 26, 2019.