Tether equilibria in proximity to a circularly orbiting satellite and their stability criteria

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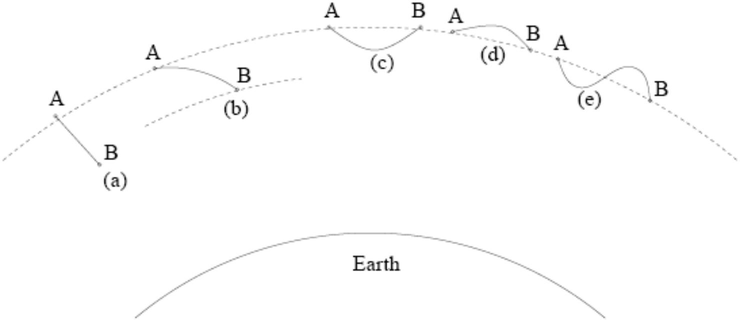


Figure 1: Examples of relative equilibria of a tether system: (a) radial, (b)-(e) wavy.

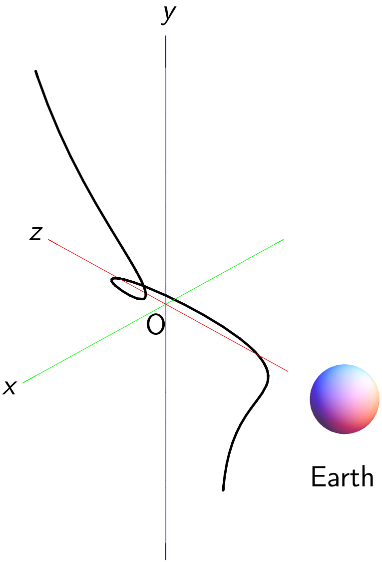
(Krupa et al. Relative equilibria of a tethered satellite systems and their stability for very stiff tethers // Dynamical systems, Volume 16, Issue 3, January 2001. P. 253-278.)

Orthogonal Orbital Coordinate System Oxyz

The linearized potential u, due to combined gravitational and centrifugal forces, in a point (x, y, z) is independent of x and is given by [3]

$$u(x, y, z) = \frac{w^2}{2}\,(y^2 - 3z^2),$$

where w is the, presumably constant, angular speed of the orbital motion.



Equilibria as conditional extremals

of tether potential

$$U = U(\gamma) = \int_{t_0}^{t_1} \left(y^2 - 3z^2\right) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

subject to constraint equation

$$L = L(\gamma) = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = l,$$

where

$$\gamma : \; t \mapsto (x, y, z),$$

satisfies boundary conditions

$$\gamma(t_0) = (x_0, y_0, z_0), \;\; \gamma(t_1) = (x_1, y_1, z_1).$$

Curvature and Torsion

We might impose upon general position solutions a natural parametrization:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 1.$$

The curvature τ_1 and torsion τ_2 are then given by:

$$\tau_1 := |\ddot{\gamma}| = \frac{\sqrt{\nu}}{|\mu|},$$

$$\tau_2 := \frac{\dot{\gamma} \cdot \left(\ddot{\gamma} \times \dddot{\gamma}\right)}{\tau_1^2} = \frac{3\left(\dot{y}z - y\dot{z}\right)\dot{x}}{\nu},$$

where

$$\mu = \frac{y^2 - 3z^2 - \lambda}{2},$$

$$\nu = y^2 + 9z^2 - (y\dot{y} - 3z\dot{z})^2 = (y\dot{z} + 3\dot{y}z)^2 + \left(y^2 + 9z^2\right)\dot{x}^2.$$

Planar configurations

The torsion identically vanishes iff

$$(y^2 + \dot{y}^2)(z^2 + \dot{z}^2)\,\dot{x} = 0$$

iff

$$(y^2 + \dot{y}^2)(z^2 + \dot{z}^2)\,\dot{x} \equiv 0,$$

corresponding to configuations in three mutually orthogonal planes:

$z \equiv 0$ – orbital plane,

$y \equiv 0$ – the plane, orthogonal to radius vector,

$x \equiv \mathrm{const}$ – the plane, orthogonal to the direction of orbital motion.

Equilibria in a constant force field

General position solutions are (up to reflection across the Ox-axis, translation and dilation) given by

$$z = c_1 \cosh\left(\frac{x}{c_1}\right), \quad c_1 > 0.$$

The the coefficient c_1 is a constant multiple of:

- the dilation coefficient
- the (imaginary) period
- the horizontal tension component

Equilibria in a linear parallel force field

Solutions are (up to reflection across and translation along the Ox-axis and dilation) given by doubly periodic functions:

$$z = \begin{cases} z_* = \sqrt{\frac{c_2}{2}\left(\mathcal{R}_\beta\left(\frac{x}{\sqrt{2c_2}}\right) + \mathcal{R}_\beta^{-1}\left(\frac{x}{\sqrt{2c_2}}\right)\right) + \lambda}, & \pm\lambda \neq c_2 > 0, \\ \frac{\sqrt{\lambda^2 - c_2^2}}{z_*}, & \lambda > c_2 > 0, \end{cases}$$

divided into classes and subclasses by periodic function:

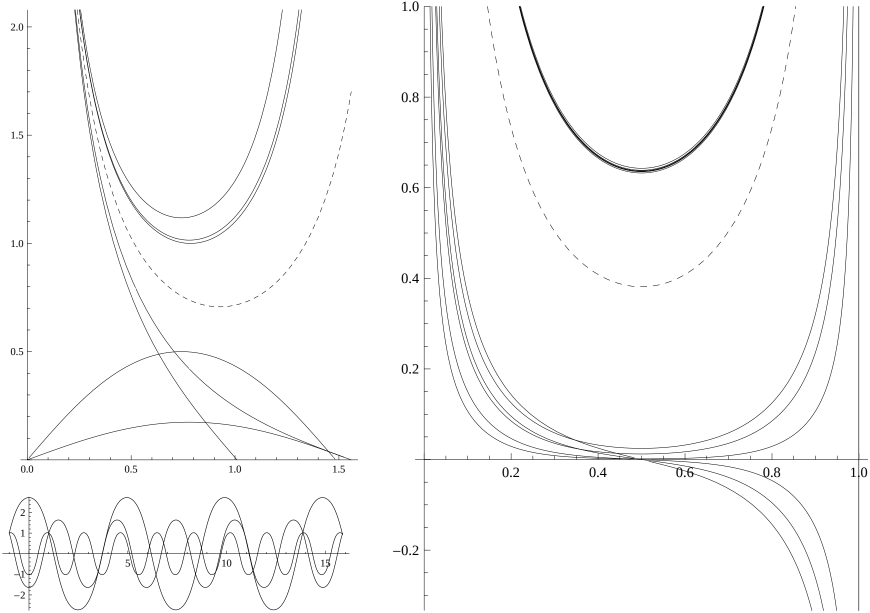
$$z = \sqrt{2\,c_2}\ \sec\left(\sqrt{\frac{2}{c_2}}\,x\right), \quad c_2 > 0,$$

vertical (parallel to Oz) and horizontal-axial (along Ox) solutions.

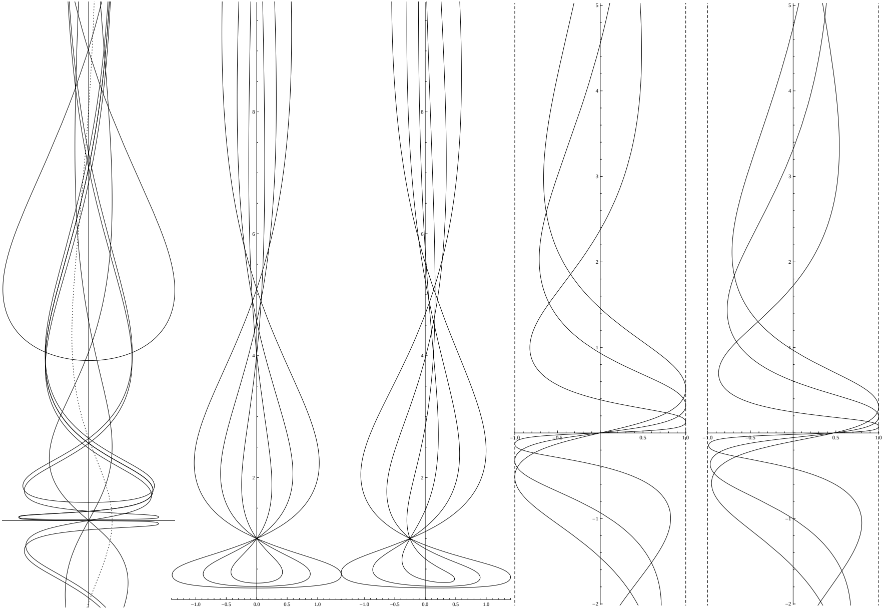
Essential elliptic function

Here R_β – is a second order elliptic function with a double pole at zero, whose values at the points where its derivative vanishes are 0, $\beta = c_2^{-1}\lambda + \sqrt{c_2^{-2}\lambda^2 - 1}$ and $\beta^{-1} = c_2^{-1}\lambda - \sqrt{c_2^{-2}\lambda^2 - 1}$. The function R_β is a fundamental elliptic functions, which is determined by a single parameter β, and differs by an additive constant from a Weierstrass elliptic function. The period parallelogram for R_β is a rectangle for $\lambda^2 > c_2^2 > 0$, and is a rhombus for $\lambda^2 < c_2^2 > 0$.

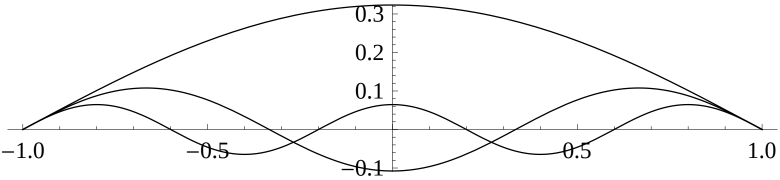
Three families of solutions



Equilibria in a plane, orthogonal to orbital motion



Pojaritsky stability investigation of Appell's solutions



Вычисление полупериодов

Вычисление полного эллиптического интеграла методом Гаусса распространяется и на случаи, когда квартический многочлен в подынтегральном выражении заменяется на кубический:

$$\int_0^1 \frac{dx}{\sqrt{x^3 + 3\alpha x^2 + x}} = \frac{\sqrt{\beta}\,\pi}{2\,M(\beta)},$$

$$\int_{-\beta}^0 \frac{dx}{\sqrt{-x^3 - 3\alpha x^2 - x}} = \frac{\sqrt{\beta}\,\pi}{M(\sqrt{1-\beta^2}\;)}.$$

и тем самым, вычисляются полупериоды u_+ и u_- эллиптической функции $\mathcal{R}_\beta$:

$$u_+ = \frac{\sqrt{\beta}\,\pi}{2\,M(\beta)}, \;\; u_- = \frac{i\sqrt{\beta}\,\pi}{2\,M(\sqrt{1-\beta^2}\;)}.$$

Вычисление значений дзета-функции Вейерштрасса в точках, соответствующих полупериодам

Теперь введём последовательность троек $\{x_m,\ y_m,\ z_m\}_{m=0}^{\infty}$:

$$x_{m+1} = \frac{x_m + y_m}{2},\ y_{m+1} = z_m + \sqrt{(x_m - z_m)(y_m - z_m)},$$

$$z_{m+1} = z_m - \sqrt{(x_m - z_m)(y_m - z_m)}.$$

Модифицированным арифметико-геометрическим средним двух положительных чисел x и y назовём общий предел последовательностей $\{x_m\}_{m=0}^{\infty}$ и $\{y_m\}_{m=0}^{\infty}$ при $x_0 = x$, $y_0 = y$ и $z_0 = 0$ и обозначим его $N(x,\ y)$. Величины $\zeta\left(u_{\pm}\right)/u_{\pm}$ могут быть также высокоэффективнно вычислены. А именно,
$\zeta\left(u_{+}\right)/u_{+} = N(1/\beta,\ \beta) - \alpha$,
$\zeta\left(u_{-}\right)/u_{-} = 1/\beta - \alpha - N(1/\beta,\ 1/\beta - \beta)$.

α-family of surfaces ($\alpha > 2/3$)

$$S : (s,t) \mapsto \left(\frac{|z(t)|}{t-s}, \frac{|z(s)|}{t-s}, \frac{l(s,t)}{t-s} \right), \ (s,\ t) \in \mathbb{R}^2,$$

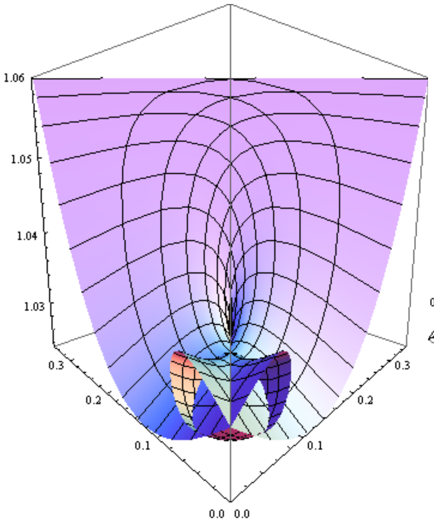
where

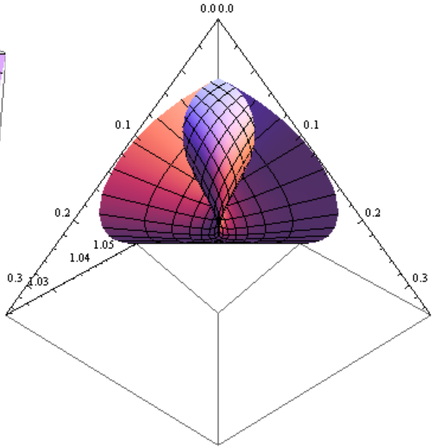
$$l(s,t) = \frac{\alpha(t-s)}{2} + \frac{1}{w_+}\Big(\zeta_*(u_+t + u_-) - \zeta_*(u_+s + u_-)\Big),$$

$$z(t) = \frac{\sqrt{3\alpha - 2}}{w_+}\ \mathrm{sn}\left(\sqrt{3\alpha+2}\ u_+t,\ \sqrt{\frac{3\alpha-2}{3\alpha+2}}\right) =$$

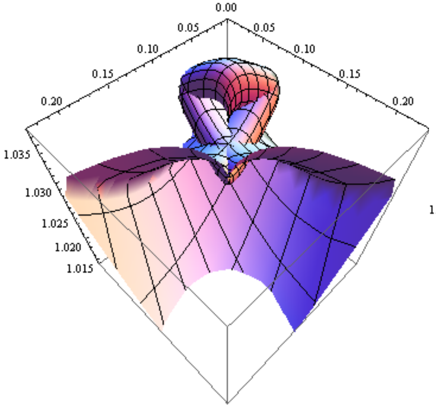
$$= \frac{1}{w_+}\sqrt{\wp_*(u_+t + u_-) + 2\alpha} = \frac{d}{w_+}\left(\sqrt{\wp_*(u_+t) + 2\alpha}\ \right)^{-1}.$$

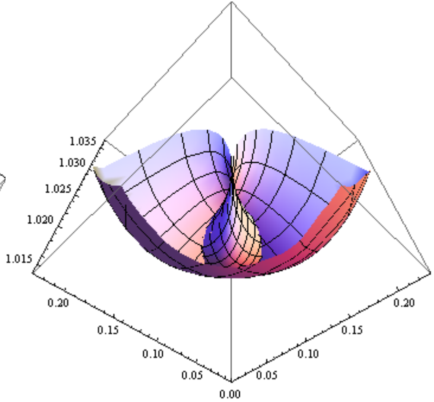
$S(0,\ 1) \times (1,\ 2),\ \alpha = 1/\sqrt{2}$



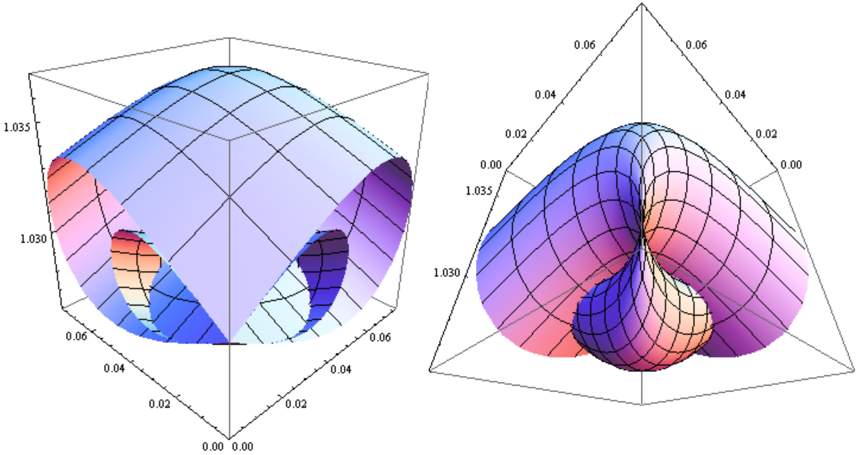


$S(0,\ 3/2) \times (3/2,\ 5/2),\ \alpha = 1/\sqrt{2}$

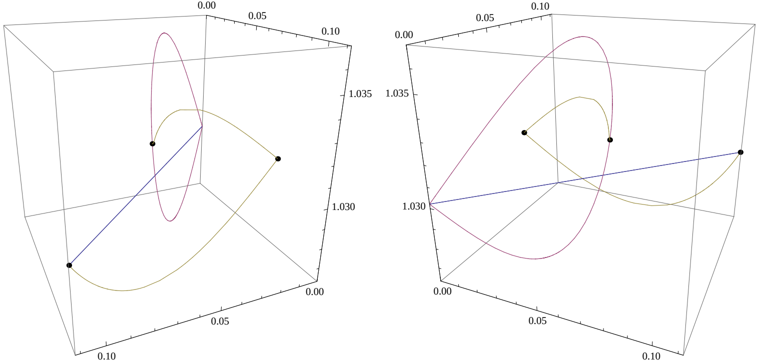




$S(0,\ 1) \times (2,\ 3),\ \alpha = 1/\sqrt{2}$



The conjugacy curve



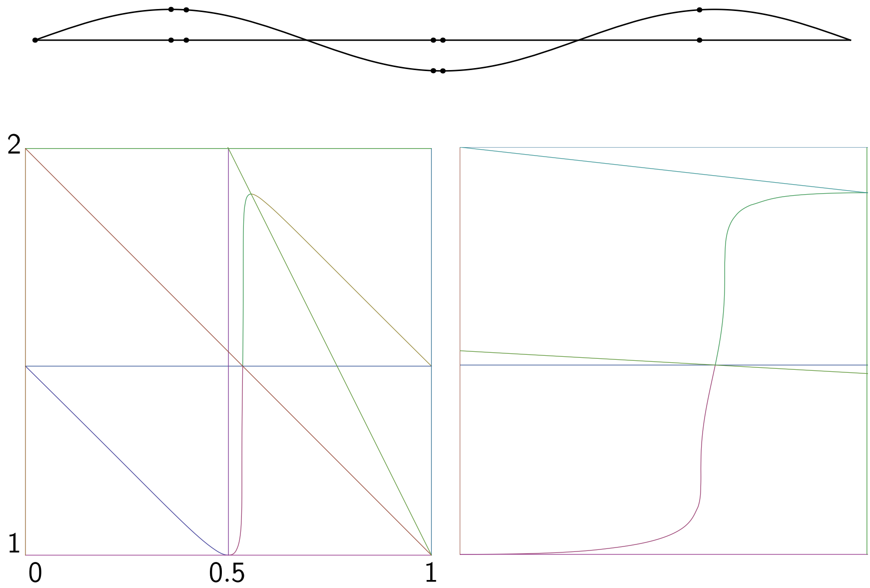
Properties of the conjugacy function

The conjugacy function $g = g_\alpha$ is periodic with period 1 and satisfies:

- $g(0) = r_\alpha,\ g(1/2) = 1,\ g(m_\alpha) = 3 - 2\,m_\alpha,$
- $g(-g(s) - s) = g(s)$
- $g'(-g(s) - s)\big(g'(s) + 1\big) + g'(s) = 0.$

In particular, $g'(1/2) = 0 = g'(m_\alpha)\,.$

A graph of the conjugacy function



ЛИТЕРАТУРА

1. Адлай С. Ф. Двойственность решений задачи о равновесии нити в притягивающих и отталкивающих полях параллельных сил // Задачи исследования устойчивости и стабилизации движения. Москва: Вычислительный Центр РАН, 2009. С. 110-118.

2. Арнольд В. И. Геометрические методы в теории обыкновенных дифференциальных уравнений. М.: Редакция журнала “Регулярная и хаотическая динамика”, 2002. 400 с.

3. Белецкий В. В. Левин Е. М. Динамика космических тросовых систем. Москва: Издательство “Наука”. 1990.

4. Пожарицкий Г. К. Устойчивость равновесий механических систем, включающих гибкую нерастяжимую нить // Прикладная математика и механика, том 37, № 4, 1973. С. 647-658.

5. Adlaj S. Tether equilibria in a linear parallel force field // 4th International Young Researchers Workshop on Geometry, Mechanics and Control, Ghent, Belgium, January 11-13, 2010. http://www.wgmc.ugent.be/adlaj.pdf (23 pages).

6. Adlaj S. An inverse of the modular invariant // Cornell University Library arXiv:1110.3274v1 [math.NT], October 14, 2011. http://arxiv.org/abs/1110.3274 (4 pages).

7. Appell P. Lacour E. Principes de la Theorie des Fonctions Elliptiques et Applications. Paris: Gauthier-Villars, 1922. 503 p.

8. Appell P. Sur une interpretation des valeurs imaginaires du temps en Mecanique // Comptes Rendus Hebdomadaires des Sceances de l’Academie des Sciences, Vol. 87, No. 1, (July) 1878.

9. Forsyth A. R. Calculus of variations. London: Cambridge University Press, 1927. 656 p.