The Inverse of the Modular Invariant j(τ)

1. Introduction

In [1], page 1 the following inverse of the modular invariant j(τ) presented at the CCRAS (Moscow, Russia) is given:

(1) $$k_0(x) = \frac{\mathrm{G}(1, \sqrt{1-x^2})\, I}{\mathrm{G}(1, x)}$$

(2) $$k_1(x) = \frac{\sqrt{x+4}}{2} - \frac{\sqrt{x}}{2}$$

(3) $$k_2(x) = \frac{3}{2}\,\frac{x}{k_3(x)} + \frac{3}{2}\,k_3(x) - 1$$

(4) $$k_3(x) = (\sqrt{x^2 - x^3} - x)^{(1/3)}$$

(5) $$k_0(k_1(k_2(j)))$$

(2.1) $$k_1(x)^2 = \frac{x}{2} + 1 - \frac{\sqrt{x\,(x+4)}}{2}$$ the square of k_1 (x)

The equation for k_3 (j) is:

(6) $$x^6 + 2\,x^3\,j + j^3$$

If x is a solution, also j / x is a solution (the – sign for the square root in (4))

The equation for k_2 (k_3 (j)) is:

(7) $$x^3 + 3\,x^2 + x\left(-\frac{27\,j}{4} + 3\right) + 1$$

The degree of this equation is only 3, because in (3) k_3 (j) and j / k_3 (j) yield the same k_2 (j) !

The equation for k_1 (k_2 (k_3 (j))) ^ 2 (formula (2.1)) is:

(8) $$1-3x+\left(6-\frac{27j}{4}\right)x^2+\left(-7+\frac{27j}{2}\right)x^3+\left(6-\frac{27j}{4}\right)x^4-3x^5+x^6$$

If x is a solution, also 1 / x is a solution (the – sign for the square root in (2.1))

The formula (8) above is equivalent to the equation for λ in formula (3.3) in [2]:

$$J(\tau) := \frac{4}{27}\frac{(\lambda^2(\tau)-\lambda(\tau)+1)^3}{\lambda^2(\tau)(\lambda(\tau)-1)^2}.$$

So (9) $k_1(j)=\sqrt{\lambda(j)}$

The well known inverse of the modular invariant e.g. in the appendix A of[2]:

$$\tau = \mathrm{i}\ \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\lambda\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\lambda\right)}.$$

So formula (1) with the inverse of the modular invariant in terms of the arithmetic – geometric mean G (1, x) follows from the well known identity:

(10) $$\frac{1}{\mathrm{G}(1,\sqrt{1-\lambda})} = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\lambda\right)$$

References

[1] S. Adlaj, An inverse of the modular invariant
Preprint arXiv:1110.3274v1 [math.NT] (14 Oct 2011)

[2] K. Vogeler & M. Flohr, Pure Gauge SU(2) Seiberg-Witten Theory and Modular Forms
Preprint, arXiv:hep-th/0607142v2 (17 Jul 2007)

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