An Arithmetic-Geometric Mean of a Third Kind!

Semjon Adlaj(✉)

Federal Research Center "Informatics and Control"
of the Russian Academy of Sciences, Vavilov St. 44, Moscow 119333, Russia
SemjonAdlaj@gmail.com

Abstract. The concept of the generalized arithmetic-geometric mean (GAGM) embraces both the arithmetic-geometric mean (AGM) and the modified arithmetic-geometric mean (MAGM) as two special concepts. The GAGM is applied for attaining a unifying formula for calculating complete elliptic integrals (CEI), including those of the third kind, thereby providing a conceptual basis for their exploration and exact evaluation, bypassing typical troubles of common software in calculating CEI. Detailed clarifying examples are provided.

Keywords: Generalized arithmetic-geometric mean · Linear fractional transformation · Quadratic convergence · Complete elliptic integral

1 Introduction

The arithmetic-geometric mean (AGM) is the key for attaining a "perfect" formula for calculating complete elliptic integrals (which we shall abbreviate as CEI whether singular or plural). The first perfect formula for calculating CEI of the first kind was obtained by Gauss. Aside from conciseness and exactness, it gave rise to an iterative sequence of intervals, swiftly converging to their common point. A termination at any step requires no additional calculations of error estimates, as other (imperfect) formulas usually require, since the exact value is guaranteed to lie inside its corresponding interval. The same process, based on Landen transformations, turned out being generalizable to calculating CEI, of any kind, via a quadratically convergent procedure. Surprisingly, however, the second perfect formula (possessing all the virtues of the first) for calculating CEI of the second kind had skipped the attention of all for over two centuries after discovering the first.[1] But only a few additional years were required to attain the third (general) perfect formula for calculating CEI of the third (or any) kind. As was the case with the two formulas, preceding it, the general formula gives rise to an iterative sequence of intervals, quadratically collapsing onto their common point. And, as before, aside from basic arithmetic operations, only a single square-root operation is required at each iteration!

[1] Leading some to allege (in desperation) that no simple exact formula for calculating the perimeter of an ellipse existed. Nevertheless, one ought not overestimate the significance of the second formula which must remain secondary to the first, without which it could not have been conceived. The two formulas "resonate" one with other, and the second, borrowing a word from [22], "echoes" the first.

M. England et al. (Eds.): CASC 2019, LNCS 11661, pp. 37–56, 2019.
https://doi.org/10.1007/978-3-030-26831-2_3

2 An Historical Overview of Elliptic Integrals

A dramatic struggle for efficiently calculating (complete and incomplete) elliptic integrals emerged with their inoculation by Fagnano.[2] Fagnano's contribution [15] to the division of elliptic arcs constitutes a most remarkable and never fading jewel of mathematics of all time![3] But it even brighter highlighted the necessity for efficiently calculating CEI, since it clarified how calculating incomplete elliptic integrals incessantly depended upon calculating CEI. A breakthrough was carried out by Gauss, who recorded the discovery of his unsurpassable arithmetic-geometric mean (AGM) method for calculating CEI of the first kind, in his diary on May 30, 1799 [21],[4] thereby laying the foundation for a distinctly novel and superb approach for calculating CEI (of any kind). Nevertheless, a formula as simple and powerful for calculating CEI of the second kind had to await December 16, 2011 to be discovered! The modified arithmetic-geometric mean (MAGM), being the necessary concept for attaining the second formula, turned (moreover) being the basic concept, underlying the generalized arithmetic-geometric mean (GAGM), which enabled on September 2, 2015 attaining the third (general) formula for calculating CEI of third (and any) kind. The generalization of MAGM to GAGM was preceded by constructing the (so-called) elliptic and coelliptic polynomials for carrying out highly efficient arithmetic on elliptic curves, including division. Earlier, on May 30, 2011, a canonical fast inverse of the modular invariant was obtained [4], further unraveling a tight relationship between the modular invariant and CEI. Fourteen new special values of the modular invariant were calculated in 2014, and an infinite family of identities, called *modular polynomial symmetries*, were first presented on April 16, 2014 at the 7$^{\text{th}}$ PCA annual conference in St. Petersburg, Russia, and subsequently represented at a seminar at Moscow State University [6]. A crucial connection between calculating the roots of the modular equation of level p and calculating the p-torsion points, on a corresponding elliptic curve, must (surprisingly) be entirely attributed to Galois. Relevant details on Galois' amazing (yet far from fully appreciated) contribution to elliptic functions (and integrals) are given in [2,4]. Certainly, the idea, involving the action of the projective linear group in the main construction of this paper was guided by Galois,[5]

[2] The (highly successful) term "elliptic integral" in and of itself was apparently invented by Fagnano.

[3] According to Fricke [16, Vorwort], the day December 23, 1751 when Euler acknowledged the receipt of Fagnano's two-volume work was regarded by Jacobi as "the birthday of the theory of elliptic functions". On January 27, 1752 Euler, crediting Fagnano, made his first presentation (to the Berlin Academy of Sciences) on the addition theorem for elliptic integrals.

[4] Strangely, Gauss' method remained either unknown or unappreciated, until recently, as pointed out in [23, Appendix O: The Simple Plane Pendulum: Exact Solution] and further explained in [5].

[5] Those overly concerned that Galois' contribution has ever been overestimated must rest assured that it was not! Up to these days, Galois' last letter [17], which he wrote on the eve of his murder May 30, 1832, remains tragically untangled in spite of all efforts of those who never underestimated it!

whose abilities, as rightfully admitted in [18, 2.21. “L’unique” – ou le don de solitude], far exceeded ours.

3 The Generalized Arithmetic-Geometric Mean

We shall reserve the letter n to denote a natural number, including zero.

3.1 Construction and Definition

The modified arithmetic-geometric sequence was presented in [3,7–9,24]. It is the recursively defined triple sequence

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := z_n + \sqrt{(x_n - z_n)(y_n - z_n)},$$

$$z_{n+1} := z_n - \sqrt{(x_n - z_n)(y_n - z_n)}.$$

Given such a sequence $\{x_n,\, y_n,\, z_n\}_{n=0}^{\infty}$, we introduce (another) recursively defined sequence of (single-valued) parametric functions:[6]

$$u_{n+1} = u_{n+1}(t) = u_{n+1}(t, c, x_0, y_0, z_0) := \frac{c_n\, u_n - y_{n+1}\, z_{n+1}}{c_n + u_n - 2\, z_n},\ c_n := u_n(c),$$

where c is a fixed real parameter and the function u_0 is (naturally) presumed to coincide with the identity function: $u_0(t) = t$. We proceed to defining the functions

$$v_n = v_n(t) = v_n(t, a, c, x_0, y_0, z_0) := \frac{t - a_n}{t - c_n},\ a_n := u_n(a),$$

$$w_n = w_n(t) = w_n(t, b, a, c, x_0, y_0, z_0) := \frac{v_n(t)}{v_n(b_n)},\ b_n := u_n(b),$$

where a and b are (also) fixed real parameters distinct from c and each other.

We shall refer to the sextuple sequence

$$\{x_n,\, y_n,\, z_n,\, a_n,\, b_n,\, c_n\}_{n=0}^{\infty}$$

as the generalized arithmetic-geometric sequence (abbreviated as GAGS whether singular or plural).[7] The sequence $\{w_n\}_{n=0}^{\infty}$ is thereby seen as a sequence of linear fractional (Möbius) transformations, generated by GAGS, successively mapping the sequence of ordered triples $(a_n,\, b_n,\, c_n)_{n=0}^{\infty}$ to the (fixed) ordered triple $(0,\, 1,\, \infty)$.

Define the generalized arithmetic-geometric mean (GAGM) of two (strictly) positive numbers x and y, for a given pairwise distinct real parameters a, b and

[6] The adjective “parametric” is meant to indicate that each such (single-valued) function (of the argument t) does “depend” upon the (fixed) values of its parameters.

[7] Thus, the GAGS is an extended modified arithmetic-geometric sequence, with twice as many terms.

c, as the (common) limit of the sequence $\{\xi_n := w_n(x_n)\}_{n=0}^{\infty}$ and the sequence $\{\eta_n := w_n(y_n)\}_{n=0}^{\infty}$ with $x_0 = x$, $y_0 = y$ and $z_0 = 0$.

Later on, we extend the domain of the parameters a, b and c to include the point at (complex) infinity, so that a, b and c might be regarded as elements of the extended real line $\mathbb{R} \cup \infty$. However, we shall always require the parameter c to lie (strictly) outside the closed interval $[x, y]$.[8]

3.2 Basic Properties

Given a linear function $l(t) = \lambda\,(t - \mu)$, $\{\lambda \neq 0, \mu\} \subset \mathbb{R}$, we define an action of the function l upon the GAGS as

$$l \cdot \{x_n, y_n, z_n, a_n, b_n, c_n\}_{n=0}^{\infty} := \{l(x_n),\, l(y_n),\, l(z_n),\, l(a_n),\, l(b_n),\, l(c_n)\}_{n=0}^{\infty}\,, \quad (1)$$

thereby inducing an action upon the sequence $\{w_n\}_{n=0}^{\infty}$, which we shall denote by $l \cdot \{w_n\}_{n=0}^{\infty} := \{l \cdot w_n\}_{n=0}^{\infty}$, where $l \cdot w_n$ is the transformation mapping the ordered triple $(l(a_n),\, l(b_n),\, l(c_n))$ to the ordered triple $(0,\, 1,\, \infty)$. One might then verify that the sequence we have defined, in (1), is indeed a GAGS![9] Furthermore, neither the sequence $\{\xi_n\}_{n=0}^{\infty}$ nor $\{\eta_n\}_{n=0}^{\infty}$ is altered by this action, that is,

$$\xi_n = l \cdot w_n\big(l(x_n)\big) = w_n(x_n),\;\; \eta_n = l \cdot w_n\big(l(y_n)\big) = w_n(y_n),$$

so that the GAGM is invariant under the action of linear functions upon the GAGS, permitting us to speak of *equivalence classes* of GAGS. So we shall say that a GAGS is equivalent to another if the GAGM is unaltered. In particular, The homogeneity degree of GAGM is zero (unlike the AGM and MAGM which are homogeneous of degree one), and we might exploit this property to extend the domain of GAGM, for fixed parameters,[10] to include (strictly) negative values of the arguments x and y. At each iteration, we might ensure the positivity of the product $(x_n - z_n)(y_n - z_n)$, before taking its square root, via acting upon the GAGS (at the required step whenever necessary) by the (constant) function -1.

We shall denote with the same letter N three functions, which we shall nevertheless distinguish by the (total) number of their arguments. The invariance of the GAGM under the action of linear functions upon the GAGS implies that four initial arguments suffice to determine the GAGM, so we designate $N(x,\, a, b, c)$ to denote the GAGM of 1 and x for parameters a, b and c.[11] Moreover, the expression

$$\Big(\frac{(b-a)\,N(x,\; a, b, c)}{b-c} - 1\Big)/(c-a),$$

[8] This requirement is necessary for the GAGM to be well defined, as we shall soon find out.

[9] Being initiated by the sextuple $\{l(x_0),\, l(y_0),\, l(z_0),\, l(a_0),\, l(b_0),\, l(c_0)\}$, so (for all indices n) we have $l(x_0)_n = l(x_n), l(y_0)_n = l(y_n), l(z_0)_n = l(z_n), l(a_0)_n = l(a_n),\, l(b_0)_n = l(b_n),\, l(c_0)_n = l(c_n)$.

[10] Generally speaking, the parameters might also be regarded as (special) arguments.

[11] An equivalence class of any GAGS might be represented by a sequence, where the initial values y_0 and z_0 are fixed at 1 and 0, respectively.

while seemingly dependent upon four arguments x, a, b and c, has x and c as its only "true" arguments. It actually depends neither upon a nor upon b. Consequently, we might define a bivariate function

$$N(x,\, c) := N(x,\, \infty, c+1, c),$$

and employ it in order to alternatively express the preceding quadrivariate function as

$$N(x,\, a, b, c) = \frac{b-c}{b-a}\Big((c-a)\, N(x,\, c) + 1\Big).$$

The latter formula extends not only to the case $c = 0$ but, as well, to the case $c = \infty$. In these two (dual) cases the GAGM "degenerates" to a (shifted) MAGM:

$$N(x,\, a, b, 0) = \frac{b}{a-b}\left(a\, N\left(\frac{1}{x}\right) - 1\right),\ N(x,\, a, b, \infty) = \frac{N(x) - a}{b-a}, \qquad (2)$$

where the (univariate) function $N(x)$ is the modified arithmetic-geometric mean of 1 and x.

The equivalence of the latter two equations reflects a special (limiting) case of the relation

$$N(x,\, a, b, c) = N\left(\frac{1}{x},\, \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right). \qquad (3)$$ [12]

3.3 Quadratic Convergence

The *difference sequence*

$$d_n := \xi_n - \eta_n = w_n(x_n) - w_n(y_n) = \frac{v_n(x_n) - v_n(y_n)}{v_n(b_n)} = s_n(x_n - y_n),$$
$$s_n := \frac{(c_n - b_n)(c_n - a_n)}{(b_n - a_n)(c_n - x_n)(c_n - y_n)},$$

depends upon all the (three) parameters a, b and c, while the ratio

$$\frac{s_{n+1}}{s_n} = \frac{c_n - z_{n+1}}{c_n - y_{n+1}} = \frac{c_n - c_{n+1}}{c_{n+1} - y_{n+1}} = \frac{c_{n+1} - z_{n+1}}{c_n - c_{n+1}}$$

[12] This relation suggests that the defining equality of the function $N(x,\, c)$ might be substituted with the equality

$$N(x,\, c) = N\left(\frac{1}{x},\, 0,\, \frac{1}{c+1}, \frac{1}{c}\right),$$

which is suitable for explicit calculation, and is extendable to the case $c = 0$ as

$$N(x,\, 0) = N\left(\frac{1}{x}\right),$$

but, unlike the quadrivariate function, the bivariate function remains undefined for $c = \infty$.

depends upon c but neither upon a nor upon b.[13] In order to show that the GAGM is well-defined we must show that the sequence $\{d_n\}_{n=0}^{\infty}$ converges to zero. We already know that it does if $c = \infty$,[14] no matter what a and b are, since the GAGM for x and y would then coincide with the MAGM of $x/(b-a)$ and $y/(b-a)$, up to an additive constant $a/(b-a)$. The case when $c = 0$ might, as well, be reduced to the case when $c = \infty$, via identity (3) or by the first of formulas (2). The case $c = z_1 = -\sqrt{x\,y}$ would imply (whatever a and b are) that $a_1 = b_1 = c_1 = z_1$, forcing a termination of the GAGS with $d_1 = s_1 = 0$. The GAGM of x and y would then coincide with the value

$$\xi_1 = \eta_1 = \frac{1}{2}\left(1 + \frac{a\,b - x\,y}{(a-b)\sqrt{x\,y}}\right).$$ [15]

The case $c < 0$ implies that $c_n < 0$ and $2\,c_{n+1} < c_n$ (for any index n), so

$$\left|\frac{s_{n+1}}{s_n}\right| < \left|\frac{c_n}{c_{n+1}} - 1\right| < 1 \Rightarrow \left|\frac{d_{n+1}}{d_n}\right| < \frac{x_{n+1} - y_{n+1}}{x_n - y_n},$$

and the GAGM would converge never (at any iteration) slower than the MAGM does, although unlike either the descending sequence $\{x_n\}_{n=1}^{\infty}$ or the ascending sequence $\{y_n\}_{n=1}^{\infty}$ neither the sequence $\{\xi_n\}_{n=0}^{\infty}$ nor the sequence $\{\eta_n\}_{n=0}^{\infty}$ is monotone.

The last case, for convergence to be considered, is the case $c > 0$. The sequence $\{c_n\}_{n=1}^{\infty}$ is then descending and, for all $n \geq 1$, $c_n > x_n$,[16] and

$$\frac{d_{n+1}}{d_n} = \frac{(c_{n+1} - z_{n+1})(x_{n+1} - x_{n+2})}{(c_n - c_{n+1})(x_n - x_{n+1})} \approx \frac{x_n - x_{n+1}}{2\,(c_n - c_{n+1})} \approx \left(\frac{x_{n-1} - x_n}{2\,(c_{n-1} - c_n)}\right)^2,$$

where the sign for approximate equality ($\approx$) must be interpreted here as an asymptotic (as n approaches infinity) equality. Consequently, the convergence is eventually (that is, asymptotically) quadratic.[17]

3.4 Alternative Calculations

The enlisted properties of GAGM enable endlessly many means of calculating it, but we shall indicate only two more. The first exploits the identity

$$N(x,\, a, b, c) = N\Big(\sigma(x,1),\, \sigma(x,a,c),\, \sigma(x,b,c),\, \sigma(x,c)\Big),$$

[13] Elementary geometric constructions, involving mutually orthogonal circles as suggested in [10], might facilitate deriving the preceding triple-equation.

[14] We are alluding to the second formula of (2).

[15] The value on the rightmost side might be obtained by applying L'Hôpital's rule to either "undeterminate" $w_1(x_1)$ or $w_1(y_1)$.

[16] The condition that c lies (strictly) outside the closed interval, bounded by x and y must not be forgotten. We need not, however, require c to lie to the left of that interval, so c_0 need not exceed x_0. In other words, the inequality $c_n > x_n$ need not apply when $n = 0$.

[17] One might note, as well, that the monotonicity of the sequences $\{\xi_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ is restored, in this ($c > 0$) case.

$$\sigma(x,y) := \sigma(x,y,y),\ \sigma(x,y,z) := \frac{\left(\sqrt{x}+y\right)\left(\sqrt{x}+z\right)}{2\,(y+z)\,\sqrt{x}},$$

which allows introducing an *abbreviated* GAGS for which $y_n = 1$ and $z_n = 0$, for all n, and

$$\{x_{n+1} = \sigma(x_n, 1), a_{n+1} = \sigma(x_n, a_n, c_n), b_{n+1} = \sigma(x_n, b_n, c_n), c_{n+1} = \sigma(x_n, c_n)\}\,. \quad (4)$$

The second introduces a *truncated* GAGS

$$\left\{x_{n+1} = \sigma(x_n,1) = \frac{\left(\sqrt{x_n}+1\right)^2}{4\sqrt{x_n}},\ c_{n+1} = \sigma(x_n,c_n) = \frac{\left(\sqrt{x_n}+c_n\right)^2}{4\,c_n\,\sqrt{x_n}}\right\},$$

for which we skip calculating a_n and b_n, but (instead) calculate the GAGM, recursively, on the basis of the identity

$$N(x,\,c) = \tau\Big(x,\,c,\,N\big(\sigma(x,1),\,\sigma(x,c)\big)\Big), \quad (5)$$

$$\tau(x,y,z) := \frac{1}{2\,y}\left(\left(\frac{y}{\sqrt{x}} - \frac{\sqrt{x}}{y}\right)\frac{z}{4} - 1\right).$$

The truncated GAGS is not suitable for calculating the GAGM in the special case $c = 0$ or $c = \infty$ when the GAGM degenerates to MAGM, as given by formulas (2), but the abbreviated GAGS serves without exceptions. In particular, we readily infer from the limit formula, with $c = \infty$,

$$N(x^2,\,a,b,\infty) = N\left(\frac{(x+1)^2}{4\,x},\,\frac{x+a}{2\,x},\,\frac{x+b}{2\,x},\,\infty\right)$$

a recursive formula for calculating MAGM:

$$N\big(x^2\big) = x\,\left(2\,N\big(f(x)^2\big) - 1\right) = 2\,f_n(x)\,N\big(f^{n+1}(x)^2\big) - \sum_{k=0}^{n} f_k(x) \approx$$

$$\approx f_n(x) - \sum_{k=0}^{n-1} f_k(x),\ \text{where}$$

$$f_n(x) := 2^n \prod_{k=0}^{n} f^k(x),\ f^{n+1}(x) := f(f^n(x)),\ f(x) := \frac{x+1}{2\sqrt{x}},\ f_0(x) = f^0(x) = x.$$

Of course, we could have defined the GAGM via the abbreviated GAGS, as given by (4), at the cost of obscuring the origin of GAGM in MAGM.

4 Calculating Three Kinds and Three Types of CEI

Assume, unless indicated otherwise, that β and γ are two positive numbers which squares sum to one: $\beta^2 + \gamma^2 = 1$.

Before we apply GAGM, to calculating CEI, we shall further extend the domain of its parameters to include complex values, and we lift any remaining doubt that the GAGM is actually a generalized AGM by observing the identity

$$N\left(\beta^2, 1-\gamma, 1-\frac{\gamma^2}{2+\gamma}, 1+\gamma\right) = N\left(\beta^2, \beta^2 + i\beta\gamma, \beta, \beta^2 - i\beta\gamma\right) = M(\beta), [18] \quad (6)$$

where $i := \sqrt{-1}$ and $M(x)$ is the AGM of 1 and x. The identity still holds if the sign of γ, which we shall refer to as *the elliptic modulus*, is flipped.[19]

4.1 Three Formulas for Calculating Three Kinds of CEI

A CEI of the first kind I_1 is defined and calculated as

$$I_1 = I_1(\gamma) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-\gamma^2\,t^2)}} = \frac{\pi}{2\,M(\beta)}. \qquad (1799.05.30)$$

A CEI of the second kind I_2 is defined and calculated as

$$I_2 = I_2(\gamma) := \int_0^1 \sqrt{\frac{1-\gamma^2 t^2}{1-t^2}} = \frac{\pi\,N(\beta^2)}{2\,M(\beta)}. \qquad (2011.12.16)$$

Both formulas (1799.05.30) and (2011.12.16) apply at $\gamma = 0$, with $I_2(0) = I_1(0) = \pi/2$. The second applies, as well, at $\gamma = 1$, with $I_2(1) = 1$, as clarified in [7,8].

A CEI of the third kind I_3 is defined and calculated as

$$I_3 = I_3(\gamma, \delta) := \int_0^1 \frac{dt}{(t^2-\delta)\,\sqrt{(1-t^2)\,(1-\gamma^2\,t^2)}} = -\frac{\pi\,\gamma^2\,N(\beta^2,\,1-\delta\gamma^2)}{2\,M(\beta)} =$$

$$= \frac{\pi\,N(\beta^2,\,\infty,\,\beta^2-\delta\gamma^2,\,1-\delta\gamma^2)}{2\,M(\beta)},\ \ \delta \in \mathbb{C}\backslash[0,1]. \qquad (2015.09.02)$$

Put $\beta_{n+1} := \sqrt{\sigma(\beta_n^2, 1)}$ with $\beta_0 = \beta$. Thereby, the recursively defined sequence $\{\beta_n\}_{n=1}^{\infty}$ converges descendingly to one, whereas the sequence $\{\gamma_n^2 := 1-\beta_n^2\}_{n=1}^{\infty}$ is a negative sequence, converging ascendingly to zero.[20] Define, recursively, the sequence

$$\delta_{n+1} := \frac{(1-\delta_n(1+\beta_n))^2}{1-\delta_n\gamma_n^2},\ \ \delta_0 = \delta.$$

[18] An equivalent GAGS has the initial values $x_0 = \beta$, $y_0 = 1/\beta$, $z_0 = 0$, $a_0 = \beta + i\,\gamma$, $b_0 = 1$, $c_0 = \beta - i\,\gamma$. Note here that if $x_0 = \beta$ and y_0 were the values at two (out of three) half-period of an essential elliptic function, as shown in [10, figures], then a_0, b_0 and c_0 are its values at three (out of six) quarter-periods.

[19] Whereas, flipping the sign of β leads to flipping the sign of $M(|\beta|)$.

[20] Alternatively, we might define the sequence of squares $\{\gamma_n^2\}_{n=0}^{\infty}$ recursively by putting $\gamma_{n+1}^2 := \sigma(\beta_n^2, -1)$ with $\gamma_0 = \gamma$.

The recursive relation (5) implies a recursive relation for I_3:

$$I_3(\gamma_n, \delta_n) = \lambda_n \left(I_1(\gamma_n) + \frac{\mu_n}{\sqrt{\beta_n}}\, I_3(\gamma_{n+1}, \delta_{n+1}) \right),$$ [21]

$$\lambda_n = \lambda(\gamma_n, \delta_n),\ \mu_n = \mu(\gamma_n, \delta_n),$$
$$\lambda(\gamma, \delta) := \frac{\gamma^2}{2\,(1-\delta\gamma^2)},\ \mu(\gamma, \delta) := \frac{\gamma^2(\gamma^2\delta^2 - 2\,\delta + 1)}{(1-\beta)^2(\delta\gamma^2 - 1)},$$

which, along with the relation $\sqrt{\beta_n}\, I_1(\gamma_n) = I_1(\gamma_{n+1})$, implies the identity

$$I_3(\gamma, \delta) = \sum_{k=0}^{n} \eta_k\, I_1(\gamma) + \eta_n\, \mu_n\, I_3(\gamma_{n+1}, \delta_{n+1}) \Big/ \sqrt{\prod_{k=0}^{n} \beta_k}, \tag{7}$$
$$\eta_n := \lambda_0 \prod_{k=1}^{n} \mu_{k-1}\, \lambda_k,$$

exhibiting that for infinitely many values of δ, satisfying (for any n) the relation $\delta_n = 1/(1-\beta_n)$, the integral I_3 would degenerate to a multiple of I_1, by the coefficient $\sum_{k=0}^{n} \eta_k$, as μ_n vanishes. Identity (7) does not apply at $\delta = 1/\gamma^2$, where I_3 would degenerate to a multiple of CEI of the second kind, by a coefficient given in the latter of formulas (10). Observe here that the equality $\delta_n = 1/\gamma_n^2$ implies that $\lambda_n = \mu_n = \infty$. Moreover, the equivalence

$$\delta_n = \frac{1}{1-\beta_n} \Leftrightarrow \delta_{n+1} = \frac{1}{\gamma_{n+1}^2}$$

holds.

The relations

$$I_3\left(\gamma, \frac{\pm 1}{\gamma}\right) = \frac{\gamma}{2}\left(\frac{\pi}{2\,(\gamma \mp 1)} \mp I_1(\gamma)\right),$$ [22]

$$I_3\left(\gamma, \frac{\gamma \pm i\,\beta}{\gamma}\right) = -\frac{\gamma}{2\,\beta}\left(\frac{(\gamma \mp i\,\beta)\,\pi}{2} \mp i\, I_1(\gamma)\right),$$

stemming from (6), would imply that for infinitely many values of δ, satisfying the relation $\delta = (\gamma \pm i\,\beta)/\gamma$ or (for any n) the relation $\delta_n^2 = 1/\gamma_n^2$, the integral I_3

[21] An analogous recursive relation for an elliptic integral of the second kind

$$I_2(\gamma) = 2\,\sqrt{\beta}\, I_2\left(\sqrt{\sigma(\beta^2, -1)}\right) - \beta\, I_1(\gamma)$$

is equivalent to formula (2011.12.16).

[22] Either the upper or the lower sign must be consistently taken throughout this or other equations in this paper.

would degenerate to a "linear combination" of the (ubiquitous) constant π and I_1.[23] Two equivalences are in order:

$$\delta_n^2 = \frac{1}{\gamma_n^2} \Leftrightarrow \delta_{n+1} = \frac{\gamma_{n+1} + i\,\beta_{n+1}}{\gamma_{n+1}},\ \delta_n = \frac{\gamma_n \pm i\,\beta_n}{\gamma_n} \Leftrightarrow \delta_{n+1} = \frac{\gamma_{n+1} - i\,\beta_{n+1}}{\gamma_{n+1}}.$$

Put

$$\delta_\pm(x) := \frac{1}{(1-\sqrt{x}\,)\left(1+x\pm\sqrt{x\,(1+x)}\right)}.$$

The preceding (primary) identities for I_3 might be applied to deriving two (secondary) identities, corresponding to $\delta_1 = 1/(1-\beta_1)$ and $\delta_1 = 1/\gamma_1$,[24] respectively:

$$I_3(\gamma,\,\delta_\pm(\beta)) = \frac{\beta-1}{4\sqrt{\beta^3}}\left((1+\beta)^2 \pm \sqrt{1+\beta}\left(1+\sqrt{\beta^3}\right)\right) I_1(\gamma),$$

$$I_3(\gamma,\,\delta_\pm(-\beta)) = \pm\sqrt{\frac{1-\beta}{\beta}}\left(\frac{\sqrt{-\beta}\pm\sqrt{1-\beta}}{2\,(1\mp\sqrt{1-\beta})}\right)^2$$

$$\left((1+\beta)\left(\beta+3\sqrt{-\beta^3}-\left(3\,(1-\beta)+4\sqrt{-\beta}\right)\left(1\mp\sqrt{1-\beta}\,\right)\right) i\,I_1(\gamma)\right.$$

$$\left.+\left(1+\sqrt{-\beta^3}\mp\sqrt{1-\beta}\;(1+\beta)\right)\pi\right).$$

4.2 An Unifying Formula for Calculating Three Types of CEI

For a given linear fractional transformation w, determined by three parameters a, b and c:

$$w(t) = w(t,a,b,c) := \frac{(b-c)(t-a)}{(b-a)\,(t-c)},\ \ \{a,b,c\} \subset \mathbb{C}\cup\infty,$$[25]

we might, as well, define a *proper* CEI I as the integral

$$I = I(\gamma,\,a,b,c) := \int_0^1 \frac{w\left(t^2\right)\,dt}{\sqrt{(1-t^2)\,(1-\gamma^2\,t^2)}}, \tag{8}$$

in which we shall distinguish three types. The *first type* would correspond to the case when the transformation w has degenerated to a constant map, the

[23] We shall avoid specifying the algebraic properties of such "linear combination", leaving this (significant) issue to other papers and, perhaps, other authors.

[24] Note that the former value (of δ_1) is negative (real) and the latter is negative imaginary.

[25] The transformation w need not necessarily be Möbius transformation, since degenerate transformations are not (yet) excluded. In other words, the transformation w need not be a conformal automorphism of either the extended or unextended complex plane, and its determinant $(a-b)(b-c)(c-a)$ is allowed to vanish, be finite or infinite.

second type would correspond to the case when w is a linear function,[26] whereas the *third type* would correspond to the case when w is a linear fractional transformation which does not fix the point at (complex) infinity. Note, however, that the restriction upon c to be distinct from ∞ does not preclude a CEI from degenerating to a CEI of the first or the second type as are the instances

$$I\left(\gamma,\, a, b, \frac{1}{1-\beta}\right) = \frac{(1-b\,(1-\beta))\,(1-a\,(1+\beta))\; I_1(\gamma)}{2\,(b-a)\,\beta}, \tag{9}$$

$$I\left(\gamma,\, a, b, \frac{1}{\gamma^2}\right) = \frac{1-b\,\gamma^2}{(b-a)\,\gamma^2}\left(\frac{\left(1-a\,\gamma^2\right)\, I_2(\gamma)}{\beta^2} - I_1(\gamma)\right).$$

In particular, the two special values

$$I\left(\gamma,\, \infty, \frac{2-\beta}{1-\beta}, \frac{1}{1-\beta}\right) = -\frac{\gamma^2\, I_1(\gamma)}{2\,\beta},$$

$$I\left(\gamma,\, \infty, \frac{\gamma^2+1}{\gamma^2}, \frac{1}{\gamma^2}\right) = -\left(\frac{\gamma}{\beta}\right)^2 I_2(\gamma) \tag{10}$$

coincide with the values of I_3 if evaluated at $\delta = 1/(1-\beta)$ and $\delta = 1/\gamma^2$, respectively. The former of formulas (10) is, in fact, a special (first) case of identity (7).[27]

Whatever the type of I, as defined in (8), we might calculate it directly as

$$I(\gamma,\, a, b, c) = \frac{\pi\, N(\beta^2,\, 1-a\,\gamma^2,\, 1-b\,\gamma^2,\, 1-c\,\gamma^2)}{2\,M(\beta)},\text{[28]} \quad c \in \mathbb{C}\backslash[0,1] \tag{11}$$

so that the case where $a=\infty$, $b=1+\delta$, $c=\delta$ is seen as the special case where I coincided with I_3. Identity (6) might now be translated to an identity for π:

$$I\left(\gamma,\, \pm\frac{1}{\gamma}, \frac{1}{2\pm\gamma}, \mp\frac{1}{\gamma}\right) = I\left(\gamma,\, \frac{\gamma\mp i\beta}{\gamma}, \frac{1}{1+\beta}, \frac{\gamma\pm i\beta}{\gamma}\right)$$

$$= -I\left(\gamma,\, \frac{\gamma\pm i\beta}{\gamma}, \frac{1}{1-\beta}, \frac{\gamma\mp i\beta}{\gamma}\right) \equiv \frac{\pi}{2}. \tag{12}$$

The identity is extendable (for all involved integrals) to the (limit) value of the elliptic modulus γ at 0, as well as, it is extendable for the first integral taken with upper signs to the (limit) value at $\gamma=1$, that is,

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \int_0^1 \frac{2\,dt}{1+t^2} \equiv \int_0^1 \frac{(1+\gamma)\left(1-\gamma\,t^2\right)\,dt}{(1+\gamma\,t^2)\sqrt{(1-t^2)\,(1-\gamma^2\,t^2)}},\ \gamma\in(0,1).$$

---

[26] Recall that the case $c=\infty$ was not excluded.

[27] Yet, even this (first) special case, where I_3 degenerates to (a multiple of) I_1 for $\delta = 1/(1-\beta)$, seems missing from standard sources on elliptic integrals.

[28] An equivalent GAGS leading to the GAGM that appears in the numerator has the initial values $x_0=0$, $y_0=-1$, $z_0=-1/\gamma^2$, $a_0=-a$, $b_0=-b$, $c_0=-c$.

However, the second and the third integrals, in identity (12), are discontinuous at $\gamma = 1$. The upper signs correspond to the value $i\pi/2$, whereas the lower signs correspond to the value $-i\pi/2$.

We emphasize the methodological significance of a clear unifying formula (11) for calculating CEI (of any type). "The Handbook of Mathematical Functions" [1, ch. 17] fell short of accomplishing that task, as the section on "The Process of the Arithmetic-Geometric Mean" was not extended to calculating elliptic integrals of the third kind, which were left to appear in the next section of the chapter on "Elliptic Integrals" by Milne-Thomson. The current version of the latter chapter, written by Carlson [12], is amended with an expressions for calculating CEI of the third type, via AGM, in the section on "Quadratic Transformations", essentially providing yet another (as perfect) alternative for calculating the sequence ξ_n, converging to GAGM.[29] An enlightening succinct review of CEI is given in [20]. "Wolfram Mathematica" warns, in [25], that "more so than for other special functions, you need to be very careful about the arguments you give to elliptic integrals and elliptic functions" but exhibits insufficient care in evaluating the integral (8), where non-vanishing imaginary parts (occasionally!) appear for real parameters. A sample "notebook", exposing this and other typical troubles in calculating CEI, by "Mathematica 10.3",[30] is appended to this article.

4.3 The Formula for Calculating the Complementary CEI

The complementary CEI (denote by J) might as readily be calculated:

$$J = J(\gamma,\, a, b, c) := \int_1^{1/\gamma} \frac{w\big(t^2\big)\ dt}{\sqrt{(t^2-1)\,(1-\gamma^2\,t^2)}} = \frac{\pi\, N(1/\gamma^2,\, a,\, b,\, c)}{2\, M(\gamma)}, \qquad (13)$$

$$c \in \mathbb{C}\backslash[1,\, 1/\gamma].$$

The integral J, as was the case with I, would also degenerate to a CEI of the first or second type if w is, respectively, constant or linear. Furthermore,

$$J\left(\gamma,\, a, b, -\frac{1}{\gamma}\right) = \frac{1}{2}\left(1 + \frac{a\, b\, \gamma^2 - 1}{(a-b)\,\gamma}\right) I_1(\beta),\ J(\gamma,\, a, b, 0) = \frac{b\,(a\, I_2(\beta) - I_1(\beta))}{a-b},$$

and, in particular,

[29] Each sequence element ξ_n is represented by a partial sum, as was the case with the (original) expression for calculating CEI of the second kind (given, as well, in the preceding chapter by Milne-Thomson). These expressions, involving infinite sums, do (most importantly) provide quadratically convergent procedures but, unlike the (first) formula for calculating CEI of the first kind, they do not produce a sequence of intervals, providing both (lower and upper) bounds.

[30] That version of "Mathematica" was released on October 15, 2015.

$$J\left(\gamma,\,\infty,\frac{1}{\gamma},-\frac{1}{\gamma}\right)=I_1(\beta),\text{[31]}\;\;J(\gamma,\,\infty,1,0)=I_2(\beta). \tag{14}$$

We rewrite the latter special case, with $c=0$, explicitly as

$$\int_1^{1/\gamma}\frac{dt}{t^2\sqrt{(t^2-1)\,(1-\gamma^2\,t^2)}}=\frac{\pi\,N(\gamma^2)}{2\,M(\gamma)},$$

in order to emphasize that it was not excluded.[32]

5 Few Explicit Calculations of CEI via GAGM

Before we move on to numerical examples, we explicitly write down the iterative step for generating a (next) sextuple of the GAGS. It must be preceded by calculating the (temporary) values $r_2=(x_n-z_n)(y_n-z_n)$, $r_1=\sqrt{r_2}$, $t_2=z_n^2-r_2$, $t_1=2\,z_n-c_n$. Then

$$(x_{n+1},\,y_{n+1},\,z_{n+1},\,a_{n+1},\,b_{n+1},\,c_{n+1})=$$
$$=\left(\frac{x_n+y_n}{2},\,z_n+r_1,\,z_n-r_1,\,\frac{c_n\,a_n-t_2}{a_n-t_1},\,\frac{c_n\,b_n-t_2}{b_n-t_1},\,\frac{c_n^2-t_2}{c_n-t_1}\right).$$

At the terminal step, one calculates

$$v_n(b_n)=\frac{b_n-a_n}{b_n-c_n},\;v_n(x_n)=\frac{x_n-a_n}{x_n-c_n},\;v_n(y_n)=\frac{y_n-a_n}{y_n-c_n},$$
$$(\xi_n,\,\eta_n)=\left(\frac{v_n(x_n)}{v_n(b_n)},\,\frac{v_n(y_n)}{v_n(b_n)}\right).$$

Alternatively, one calculates the (same) values $(\xi_n,\,\eta_n)=(w_n(x_n),\,w_n(1))$ as they emerge from an equivalent abbreviated GAGS,[33] as given by (4), although (as we know) the transformation w_n in and of itself is not invariant under linear actions upon the GAGS.

Now, we shall presume that $\beta=\gamma=1/\sqrt{2}$. Denote, for brevity, the values $M(\sqrt{2})$ and $N(2)$ by M and N, respectively, and put

$$L:=\frac{\pi}{M}\approx 2.62205755429211981046.\text{[34]}$$

[31] Thus,

$$J\left(\gamma,\,\frac{1}{\gamma},\frac{1}{2+\gamma},-\frac{1}{\gamma}\right)=0.$$

Note that the arguments of the (complementary) integral J coincide with the arguments of the first integral I from identity (12), taken with the upper signs.

[32] The inclusion of this case ($c=0$) could not have been made possible had we chosen the conventional definition of the CEI of the third kind.

[33] Recall that for an abbreviated GAGS, $y_n=1$ for all n.

[34] Assuming π is known with sufficient precision, the precision of the latter calculation is attained after four iterations towards the value of the constant M.

semjonadlaj@gmail.com

The constant L was referred to, in [7,8], as *the lemniscate constant*. It is the semi-length of the lemniscate of Bernoulli which focal distance is $\sqrt{2}$.[35]

Firstly, we calculate the first (exceptional) case $\delta = 1/(1-\beta) = 2+\sqrt{2}$ of formula (2015.09.02), via applying (9) or the first of equations (10),

$$\int_0^1 \frac{dt}{\left(t^2-2-\sqrt{2}\,\right)\sqrt{\left(1-t^2\right)\left(1-t^2/2\right)}} = I\left(\frac{1}{\sqrt{2}},\,\infty,\,3+\sqrt{2},\,2+\sqrt{2}\right)$$
$$= -I_1\left(\frac{1}{\sqrt{2}}\right)/\sqrt{8} = -\frac{L}{4}.$$

Secondly, we calculate two "mutually" complementary CEI, which share the same absolute value

$$J\left(\frac{1}{\sqrt{2}},\,\infty,\,1,\,0\right) = -\,I\left(\frac{1}{\sqrt{2}},\,\infty,\,3,\,2\right) = I_2\left(\frac{1}{\sqrt{2}}\right),$$ [36]

where the first integral might be calculated via applying the second formula of (14), while the second integral might be calculated via applying the second formula of (10). The absolute values of both integrals turn out to coincide with the value of CEI of the second kind, which might be further evaluated as

$$I_2\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi\,N}{2\sqrt{2}\,M} = \frac{L+M}{2\sqrt{2}} \approx 1.3506438810476755025.$$ [34]

In other words, the absolute value of either of the aforesaid integrals coincides with the ratio of the semi-length of the perimeter of the self-complementary ellipse, as defined in [7,8], to the length of its diameter (that is, its major axis). The relationship of this ratio with the afore-defined constants M and L stems from the (central) case of Legendre relation, which was presented by Euler to the St. Petersburg Academy of Sciences on September 4, 1775 [14].[37] Here, we

[35] Such lemniscate is inscribed in a cocentered unit circle, as shown in [7,8, Fig. 2].

[36] The expression on the leftmost side is attained by applying formula (13) to the integral J and formula (11) to the integral I. Formula (2015.09.02) also applies at (the exception case) $\delta = 1/\gamma^2 = 2$.

[37] Another remarkable date when the first of two key ideas behind the "Gauss-Euler algorithm" was presented. Note that the combination of these two outstanding names is (nevertheless) as exceptionally rare as to require no further specification of the algorithm for calculating the constant π. Strangely, a few still argue that the term "Brent-Salamin algorithm" is preferable, being (as it seems to them) less ambiguous, "since" both names Brent and Salamin are much less frequently heard (than either Euler or Gauss). These few, including Brent [11], seem unaware that the frequency with which either the name Euler or Gauss is (separately) associated with so many methods does not imply that the two names (together) must be nearly as frequently associated with any other (or same) methods. In fact, Gauss-Euler algorithm is never confused with any other algorithm (whether or not related to calculating π), so there is no ambiguity here to be lessened.

semjonadlaj@gmail.com

might pause to express this beautiful relation with a marvelously simple and powerful formula

$$\pi = \frac{M^2}{N-1},\text{[38]}$$

giving rise to a quadratically convergent algorithm for calculating π.[39] Such formula differs radically from power series representations of π. Combining iterations we might attain convergence to an arbitrarily high order, whereas no methods exist to *accelerate* a given linearly convergent algorithm to an algorithm which order of convergence (strictly) exceeds one.[40]

Thirdly, we calculate the CEI

$$I_3\left(\frac{1}{\sqrt{2}}, -1\right) = \frac{\pi\, N(2,\, 0, 1, 2/3)}{\sqrt{2}\, M} \approx 1.273127366749682458.$$

The precision of the last approximation is attained after the fifth iteration towards $N(2,\, 0, 1, 2/3)$ (assuming that π and M are known with sufficient precision). We list "chopping-off digits" approximations for the corresponding elements of GAGS:

$$x_1 = \frac{3}{2},\ y_1 = \sqrt{2},\ z_1 = -\sqrt{2},\ a_1 = 3,\ b_1 = \frac{8}{5},\ c_1 = \frac{11}{6},$$

$$\begin{aligned}
x_2 &\approx 1.457106781186547524400844362104849039284835 9,\\
y_2 &\approx 1.456786383137055103978062198817207626803368 7,\\
z_2 &\approx -4.285213507883245201581439647236603783942712 4,\\
a_2 &\approx 1.532629576631617159351843766662252142108039 6,\\
b_2 &\approx 1.465398442160606356419084365632672998134987 4,\\
c_2 &\approx 1.478616338216314373238181397493692078838588 7,
\end{aligned}$$

$$\begin{aligned}
x_3 &\approx 1.456946582161801314189453280461028333044102 3,\\
y_3 &\approx 1.456946579927125936614834227297327115962694 9,\\
z_3 &\approx -10.027373595693616339777713521770534683848119,\\
a_3 &\approx 1.457088185743057121271957724490961274931321 0,\\
b_3 &\approx 1.456962486022700138422156262410446563383900 0,\\
c_3 &\approx 1.456987314858313193929892020973753355901702 1,
\end{aligned}$$

[38] This formula made its début in [8].

[39] Note that evaluating the square root at each iteration is best done via the quadratically convergent (so-called) Heron's method, which amounts to iteratively replacing a given approximation r of a square root of s by the arithmetic mean of r and s/r.

[40] For example, the Chudnovsky famously fast formula, for calculating π, converges (still) linearly [13].

$$x_4 \approx 1.4569465810444636254021437538791777245033986,$$ [41]

$$\begin{aligned}
y_4 &\approx 1.4569465810444636253477894912161889487201529,\\
z_4 &\approx -21.511693772431696304903216534757258316416392,\\
a_4 &\approx 1.4569465812955909691425417509597958735013842,\\
b_4 &\approx 1.4569465810726702938839128147996788375690002,\\
c_4 &\approx 1.4569465811167028907128642667070835586747654,
\end{aligned}$$

$$\begin{aligned}
x_5 &\approx 1.4569465810444636253749666225476833366117757,\\
y_5 &\approx 1.4569465810444636253749666225476833366117596,\\
z_5 &\approx -44.480334125907856235181399692062199969444545,\\
a_5 &\approx 1.4569465810444636253753615361024413575571484,\\
b_5 &\approx 1.4569465810444636253750109793093234100922159,\\
c_5 &\approx 1.4569465810444636253750802233393765414321542,
\end{aligned}$$

as well as, approximations for the corresponding elements of the difference sequence:

$$d_1 \approx 0.119398062518129278742,\ d_2 \approx 0.007245988895557086620,$$

$$d_3 \approx 0.000026834417169799896,\ d_4 \approx 0.000000000368037706275,$$

$$d_5 \approx 0.000000000000000000069.$$

The GAGM is contained in the open interval (η_5, ξ_5), where

$$\xi_5 \approx 0.686664556900553064232,\ \eta_5 \approx 0.686664556900553064163.$$

The same difference sequence and the same open interval, containing GAGM, arises had we calculated the abbreviated equivalent GAGS:

$$x_1 = \frac{4+3\sqrt{2}}{8},\ a_1 = \frac{2+3\sqrt{2}}{4},\ b_1 = \frac{5+4\sqrt{2}}{10},\ c_1 = \frac{12+11\sqrt{2}}{24},$$

$$\begin{aligned}
x_2 &\approx 1.0000557990344084608909536718021882886851740,\\
a_2 &\approx 1.0132084978986451044553767711278338098480210,\\
b_2 &\approx 1.0014998361523864412955349417792705697154935,\\
c_2 &\approx 1.0038018034645730876221835149885411210608887,
\end{aligned}$$

$$\begin{aligned}
x_3 &\approx 1.0000000001945849073694805774440659743370935,\\
a_3 &\approx 1.0000123303612025542191609858469557610928674,\\
b_3 &\approx 1.0000013850271788806221085830365503548205435,\\
c_3 &\approx 1.0000035470041381927545950431601339276865500,
\end{aligned}$$

[41] The values x_1 through x_4 were calculated earlier (with lesser precision) in [8] as successive approximations of N.

$$x_4 \approx 1.0000000000000000000023664553855388570440700,$$
$$a_4 \approx 1.0000000000109334875695742133189157444409144,$$
$$b_4 \approx 1.0000000000012280512952458860229688472649685,$$
$$c_4 \approx 1.0000000000031451258870071434118165041864399,$$

$$x_5 \approx 1.0000000000000000000000000000000000000000003,$$
$$a_5 \approx 1.0000000000000000000000085967986933596017383,$$
$$b_5 \approx 1.0000000000000000000000009655939785168947160,$$
$$c_5 \approx 1.0000000000000000000000024729542094121398816.$$ [42]

The truncated GAGS does not require calculating a_n and b_n. Instead, the transformation w_5 might be calculated, recursively, as

$$w_5(t) = \tau\left(x_0, c_0,\ \tau\left(x_1, c_1,\ \tau\left(x_2, c_2,\ \tau\left(x_3, c_3,\ \tau\left(x_4, c_4,\ \frac{1}{t - c_5}\right)\right)\right)\right)\right),$$

and so $\eta_5 = w_5(1)$ and $\xi_5 = w_5(x_5)$, where the value x_5 (and the transformation w_5) is the same whether the GAGS is abbreviated or truncated.

Whatever the case, one ought not confuse the definition of GAGM with the chosen method for calculating it. On the other hand, one must never forget that the GAGM might be calculated "independently" of the AGM.[43]

6 Conclusion

The concept of MAGM enables a "perfect" formula for calculating CEI of the second kind, as given by (2011.12.16), where a function of single variable (the elliptic modulus) appears in its numerator. A "perfect" formula for calculating CEI of the third kind requires constructing a bivariate function. Such function is constructed by "extending" the concept of MAGM to GAGM, and the formula for calculating CEI of the third kind is given by (2015.09.02). Moreover, the concept of GAGM permits constructing a quadrivariate function which is necessary for a general "perfect" formula for calculating any proper CEI, as given by (11).

Acknowledgment and Notification. The author supports an unrestricted access to knowledge, and grants his permission for using his algorithms and formulas to persons and non-profit-seeking organizations. Profit-seeking organizations, including commercial software companies and their representatives, must address the author for an explicit written permission, without which they are never permitted to use any formulas, algorithms or methods based on the concept of MAGM or GAGM.

[42] In our example, not only x_n, but a_n, b_n and c_n also converge to 1.

[43] So, of course, is the case with the MAGM which might be calculated, without the AGM, as for determnining the length of a thread in a linear parallel repelling force field [3,9].

A Worksheet on Typical Troubles with Calculating CEI

Printed from the Complete Wolfram Language Documentation, and addended to the article "An arithmetic-geometric mean of a third kind!" by S. Adlaj

(* Sample problems in exact and numerical evaluations of elliptic integrals by "Mathematica 10.3" *)

(* "Mathematica 10.3" is unable to recognize that the following elliptic integral as identically zero *)

$$\texttt{I1[k\_]} = \int_0^1 \frac{\frac{(1+k)\,t^2-1}{(1-k)\,t^2-1}\,dt}{\sqrt{(1-t^2)\,(1-(1-k^2)\,t^2)}};$$

(* An approximation at $k = 1/\sqrt{2}$ **is *)** $\texttt{N}\left[\texttt{I1}\left[1/\sqrt{2}\right]\right]$

NIntegrate::ncvb:
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = {0.999999999999999833552297401554899705442790134545403187354907677264}. NIntegrate obtained 2.366162821232365`\*^-15 and 4.2827226680842307`*^-10 for the integral and error estimates. >>

2.36616×10^{-15}

(* Neither it is able to recognize the following elliptic integrals as equivalent to an elliptic integral of the second kind *) $\left\{\int_1^{\frac{1}{k}} \frac{k^2\,t^2\,dt}{\sqrt{(t^2-1)\,(1-k^2\,t^2)}},\ \int_1^{\frac{1}{k}} \frac{dt}{t^2\sqrt{(t^2-1)\,(1-k^2\,t^2)}}\right\};$

$$\texttt{NIntegrate}\left[\frac{t^2}{2\sqrt{(t^2-1)\,(1-t^2/2)}},\ \left\{t, 1, \sqrt{2}\right\},\ \texttt{WorkingPrecision} \to 36\right]$$

NIntegrate::ncvb: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = {1.4142135623730942003299277660615939758975378987916786520544997278778704162408871717061}. NIntegrate obtained 1.3506438810476755025201659822471160697182866641263270648893709152790751355061145585697`86. and 2.571375717923436839774279633792891625430816941827454509177688212084834091567201156289`86.*^-16 for the integral and error estimates. >>

1.35064388104767550252016598224711607

$$\texttt{NIntegrate}\left[\frac{1}{t^2\sqrt{(t^2-1)\,(1-t^2/2)}},\ \left\{t, 1, \sqrt{2}\right\},\ \texttt{WorkingPrecision} \to 69\right]$$

NIntegrate::ncvb: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = {1.0000235251787573510657049851697824592345237750142292065417528660505491620748702709807729996958076317480295755259869434}. NIntegrate obtained 1.3506438810476755025197123507391015977110729509744161764711042365568627519721188826827742197396961861628748882328470889`119. and 2.2896705902782230456502227592670356762869622238669805148703894389679104324846202543975623724941165872811615939445031806`119.*^-14 for the integral and error estimates. >>

1.35064388104767550251971235073910159771107295097441617647110423655686

(* Both integrals might be accurately evaluated at $k = 1/\sqrt{2}$ **as *)**

$\texttt{N}\left[\texttt{EllipticE}\left[\frac{1}{2}\right], 71\right]$

1.3506438810476755025201747353387258413495223669243545453232537088578779

(* Things do not necessarily get any better if "Mathematica 10.3" comes up with an exact evaluation of a real-valued elliptic integral, say, the integral *)

$$\texttt{I3} = \int_0^1 \frac{t^2\,dt}{(t^2+1)\sqrt{(1-t^2)\,(1-t^2/4)}}$$

$$\frac{1}{60}\left(80\,i\sqrt{3} + 40\,\texttt{EllipticF}\left[\texttt{ArcCsc}\left[\sqrt{\frac{2}{3}}\right], \frac{8}{9}\right] + 60\,\texttt{EllipticK}\left[\frac{1}{4}\right] - 30\sqrt{2}\,\texttt{EllipticK}\left[\frac{9}{8}\right] + (9-3\,i)\sqrt{2}\,\texttt{EllipticPi}\left[\frac{9}{10} - \frac{3\,i}{10}, \frac{9}{8}\right] + (9+3\,i)\sqrt{2}\,\texttt{EllipticPi}\left[\frac{9}{10} + \frac{3\,i}{10}, \frac{9}{8}\right] - (12-4\,i)\,\texttt{EllipticPi}\left[\frac{4}{5} - \frac{4\,i}{15}, \texttt{ArcCsc}\left[\sqrt{\frac{2}{3}}\right], \frac{8}{9}\right] - (12+4\,i)\,\texttt{EllipticPi}\left[\frac{4}{5} + \frac{4\,i}{15}, \texttt{ArcCsc}\left[\sqrt{\frac{2}{3}}\right], \frac{8}{9}\right]\right)$$

(* where a non-vanising imaginary part appears! *)
N[I3]

$2.49522 + 2.3094\,i$

semjonadlaj@gmail.com

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Applied Mathematics Series, vol. 55, 10th Printing. National Bureau of Standards, Washington (1972)
2. Adlaj, S.: Galois elliptic function and its symmetries. In: Vassiliev, N.N. (ed.) 12th International Conference on Polynomial Computer Algebra, pp. 11–17. St. Petersburg department of Steklov Institute of Mathematics (2019)
3. Adlaj, S.: Thread Equilibrium in a Linear Parallel Force Field. LAP LAMBERT, Saarbrucken (2018). (in Russian)
4. Adlaj, S.: On the second memoir of Évariste Galois' last letter. Comput. Tools Sci. Educ. **4**, 5–20 (2018)
5. Adlaj, S.: An analytic unifying formula of oscillatory and rotary motion of a simple pendulum (dedicated to 70th birthday of Jan Jerzy Slawianowski). In: Mladenov, I., Ludu, A., Yoshioka, A. (eds.) Proceedings Intereational Conference on "Geometry, Integrability, Mechanics and Quantization" 2014, pp. 160–171. Avangard Prima, Sofia, Bulgaria (2015)
6. Adlaj, S.: Torsion points on elliptic curves and modular polynomial symmetries. Presented at the joint MSU-CCRAS Computer Algebra Seminar on September 24, 2014. http://www.ccas.ru/sabramov/seminar/lib/exe/fetch.php?media=adlaj140924.pdf
7. Adlaj, S.: A perfect formula for the perimeter of an ellipse. Math. Yilin **001**, 30–37 (2013). (in Chinese)
8. Adlaj, S.: An eloquent formula for the perimeter of an ellipse. Not. AMS **59**(8), 1094–1099 (2012)
9. Adlaj, S.: Mechanical interpretation of negative and imaginary tension of a tether in a linear parallel force field. In: Selected Works of the International Scientific Conference on Mechanics "Sixth Polyakhov Readings", pp. 13–18. Saint-Petersburg State University, Saint-Petersburg (2012)
10. Adlaj, S.: Eighth Lattice Points, arXiv:1110.1743 (2011)
11. Brent, R.: Old and new algorithms for pi. Letters to the Editor. Not. AMS **60**(1), 7 (2013)
12. Carlson, B.C.: Chapter 19: Elliptic Integrals. NIST Digital Library of Mathematical Functions (v. 1.0.10 released 2015–08-07). http://dlmf.nist.gov/19
13. Chudnovsky, D.V., Chudnovsky, G.V.: The computation of classical constants. Proc. Nat. Acad. Sci. U.S.A. **86**(21), 8178–8182 (1989)
14. Euler, L.: De miris proprietatibus curvae elasticae sub aequatione $y = \int \frac{xxdx}{\sqrt{1-x^4}}$ contentae. Presented to the St. Petersburg Academy of Sciences on September 4, 1775. Acta Academiae Scientarum Imperialis Petropolitinae, pp. 34–61 (1782)
15. Fagnano, G.C.: Produzioni Matematiche (1, 2). Stamperia Gavelliana, Pesaro (1750)
16. Fricke, R.: Die elliptischen Funktionen und ihre Anwendungen. Zweiter Band: Die algebraischen Ausführungen. Teubner, Leipzig (1922)
17. Galois, É.: Lettre de Galois á M. Auguste Chevalier. J. de Mathématiques Pures et Appliquées **XI**, 408–415 (1846)
18. Grothendieck, A.: Récoltes et Semailles: Réflexions et témoignage sur un passé de mathématicien. http://matematicas.unex.es/~navarro/res/res.pdf
19. Gauß, C.: Arithmetisch geometrisches Mittel. Werke **3**, 361–403 (1866). Königlichen Gesell. Wiss. Göttingen

20. He, K., Zhou, X., Lin, Q.: High accuracy complete elliptic integrals for solving the Hertzian elliptical contact problems. Comput. Math. Appl. **73**(1), 122–128 (2017)
21. Klein, F.: Gauß' wissenschaftliches Tagebuch 1796–1814. Math. Ann. **57**, 1–34 (1903)
22. Lord, N.: The moment of inertia of an elliptical wire. Math. Gaz. **98**(541), 121–125 (2014)
23. Simpson, D.G.: General Physics I: Classical Mechanics (updated on January 25, 2013 and August 26, 2014). http://www.pgccphy.net/1030/phy1030.pdf
24. Modified Arithmetic-Geometric Mean. Mathematics Stack Exchange. https://math.stackexchange.com/questions/391382/modified-arithmetic-geometric-mean. Accessed 3 June 2019
25. Wolfram Language Tutorial: Elliptic Integrals and Elliptic Functions. Wolfram Language & System Documentation Center. http://reference.wolfram.com/language/tutorial/EllipticIntegralsAndEllipticFunctions.html. Accessed 3 June 2019