Duality of equilibria of a simple pendulum

Semjon Adlaj

Let $\mathscr{D}$ denote the (non-linear) differential operator

$$\mathscr{D}(\theta) := \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \cos\theta,\ \theta = \theta(t).$$

The differential equation $\mathscr{D}(\theta) = -1$ has the trivial solution $e^{i\theta} \equiv 1$, where e is the base of natural logarithms and $i := \sqrt{-1}$. On the other hand, The differential equation $\mathscr{D}(\theta) = 1$ has the trivial solution $e^{i\theta} \equiv -1$. These two trivial solutions might, respectively, be interpreted as the stable and unstable equilibrium position of a simple pendulum.

Remarkably, another elementary (yet non-trivial) solution pair to the afore indicated differential equation pair is yet to appear in mechanics textbooks. In fact,

$$t \mapsto e^{i\theta} = f(t) := \frac{\sin t + 1}{\sin t - 1} = -\left(\sec t + \tan t\right)^2$$

is a solution to the first differential equation $\mathscr{D}(\theta) = -1$, whereas

$$t \mapsto e^{i\theta} = -f(-it) = \frac{i - \sinh t}{i + \sinh t} = \left(\operatorname{sech} t + i\tanh t\right)^2$$

is a solution to the second differential equation $\mathscr{D}(\theta) = 1$. The latter solution has been called *Abrarov's solution.*

Observe that the range of Abrarov's solution, for real values of the argument t, lies on the unit circle (centred at the origin in $\mathbb{C}$), which might be identified with the configuration space of a simple pendulum. The direction of the (uniform) external force field is presumed to coincide with the positive direction of the real axis. Thus, Abrarov's solution provides the formula of motion of the pendulum, from the stable equilibrium (lowest) position of the pendulum at $t = 0$ to its unstable equlibrium (upmost) position at $t = \infty$, in the critical case when the kinetic energy vanishes at that unstable equilibrium position. Thereby, Abrarov's solution is the solution that (on the one hand) separates oscillatory mode from rotary mode of motion of a simple pendulum, and (on the other hand) it is the solution that "glues" together oscillatory and rotary modes of motion.

The modulus 2π of the (imaginary) period of Abrarov's solution coincides with the (real) period of the function f, that is, the period of the corresponding "small angle" pendulum.

A belief that "a single analytic expression is not possible for describing the trajectories of motion of the pendulum in both oscillatory and rotary zones", (incautiously) articulated by Lidov M.L. on p. 256 of his "Lecture course on Theoretical Mechanics" (2^{nd} edition. Fizmatlit, 2010), was "supported" by topological arguments and seemed convincing to "specialists", who apparently confused the absence of an unifying analytic formula of motion, in both oscillatory and rotary regimens, of a simple pendulum by the end of the second millennium with its nonexistence. That belief was refuted by formula 5 of the article "An analytic unifying formula of oscillatory and rotary motion of a simple pendulum", dedicated to Jan Jerzy Slawianowski 70^{th} birthday and made available for download at my personal web page (http://SemjonAdlaj.com/).