Determining the performance of a binary test

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Introduction and definitions

Let d denote a disease and let t denote a binary test. Define $p(d{\cdot}t)$ as the probability of being afflicted with the disease d and testing positively on the test t, $p(t\,|\,d)$ as the conditional probability of testing positively on the test t having been afflicted with the disease d, $p(d)$ as the incidence of d, that is, the unconditional probability of being afflicted with d, $p(t)$ as the unconditional probability of testing positively on t. Let D and $\bar{D}$ denote a pair of dual to each other diseases, that is, D necessarily strikes everyone free from $\bar{D}$ and never afflicts anyone afflicted with $\bar{D}$. Let T and $\bar{T}$ denote a pair of dual to each other binary tests, that is, T turns being positive if and only if $\bar{T}$ turns being negative. We regard the performance, of the test T for diagnosing the disease D, as being fully determined once we determine the four probabilities

- $p_0 = p(D \cdot T)$ – the probability of being afflicted with D and testing positively on T,
- $p_1 = p(\bar{D} \cdot T)$ – the probability of being free from D and testing positively on T,
- $p_2 = p(\bar{D} \cdot \bar{T})$ – the probability of being free from D and testing negatively on T,
- $p_3 = p(D \cdot \bar{T})$ – the probability of being afflicted with D and testing negatively on T.

Since

$$p_0 + p_1 + p_2 + p_3 = 1,$$

the four probabilities are endowed with only three degrees of freedom and all four are determined whenever any three of them are given or calculated.

It is customary, when determining the performance, of a test T for diagnosing a disease D, to be given two measures of such, named the sensitivity and the specificity, which we shall denote by the letters p and q, respectively. These are

- $p = p(T\,|\,D)$ – the probability of testing positively on the test T having been afflicted with the disease D,
- $q = p(\bar{T}\,|\,\bar{D})$ – the probability of testing negatively on the test T while being free from the disease D.

The two probabilities p and q are insufficient, of course, for determining the four probabilities p_0, p_1, p_2 and p_3. The determination of the latter four probabilities requires, in addition to p and q, another third probability independent of p and q. Such a probability could be the incidence, of the disease D, which we shall denote by r, that is,

- $r = p(D)$ – the probability of being afflicted with D.

Now, the two conditional probabilities p and q, along with the unconditional probability r, are sufficient for calculating the four probabilities p_0, p_1, p_2 and p_3. Alternatively, we may calculate the four probabilities p_0, p_1, p_2 and p_3 if given three probabilities P, Q and R dual to the, already introduced, probabilities p, q and r:

- $P = p(D\,|\,T)$ – the probability of being afflicted with D having tested positively on T,
- $Q = p(\bar{D}\,|\,\bar{T})$ – the probability of being free from D having tested negatively on T,
- $R = p(T)$ – the probability of testing positively on T.

Binding relations

The duality between the two probabilities corresponding to same letter, in small and capital size, is induced by formally swapping D with T. We may express the four probabilities p_0, p_1, p_2 and p_3 explicitly in terms of the three probabilities p, q and r, or we may express them explicitly in terms of three probabilities P, Q and R:

$$\begin{aligned} p_0 &= pr &&= PR, \\ p_1 &= (1-q)(1-r) &&= \big(1-P\big)R, \\ p_2 &= q\,(1-r) &&= Q\,\big(1-R\big), \\ p_3 &= (1-p)\,r &&= \big(1-Q\big)\big(1-R\big). \end{aligned}$$

Assume, in what follows, the strict inequality

$$0 < p_0 p_1 p_2 p_3. \tag{1}$$

The four probabilities p_0, p_1, p_2 and p_3 are, in fact, determined by any three of the six probabilities p, q, r, P, Q and R. In other words, there are only three degrees of freedom bestowed upon any four of the six probabilities p, q, r, P, Q and R. In particular, there are only three degrees of freedom bestowed upon the four probabilities p, q, P and Q. The relation binding the latter four probabilities is

$$\Big(p^{-1}-1\Big)\Big(q^{-1}-1\Big) = \Big(P^{-1}-1\Big)\Big(Q^{-1}-1\Big). \tag{2}$$

There are 8 four-element-subsets, containing either r or R, of the six-element-set $\{p, q, r, P, Q, R\}$. There are 8 corresponding binding relations – one for each such subsets. Thus, both r and R may be explicitly expressed via any three of the four probabilities p, q, P and Q:

$$\begin{aligned} r^{-1} - 1 = \frac{p\Big(P^{-1}-1\Big)}{1-q} = \frac{1-p}{q\Big(Q^{-1}-1\Big)} = p\Big(P^{-1}-1\Big) + \frac{1-p}{Q^{-1}-1} = \\ = \left(q\Big(Q^{-1}-1\Big) + \frac{1-q}{P^{-1}-1}\right)^{-1}, \end{aligned} \tag{3}$$

$$R^{-1}-1=\frac{P\Big(p^{-1}-1\Big)}{1-Q}=\frac{1-P}{Q\Big(q^{-1}-1\Big)}=P\Big(p^{-1}-1\Big)+\frac{1-P}{q^{-1}-1}=$$
$$=\left(Q\Big(q^{-1}-1\Big)+\frac{1-Q}{p^{-1}-1}\right)^{-1}. \tag{4}$$

The 4 binding relations, corresponding to the 4 four-element-subsets containing both r and R, along with either p or q, and either P or Q, are:

$$\begin{aligned} R^{-1}r &= p^{-1}P, \\ R^{-1}\big(1-r\big) &= \big(1-q\big)^{-1}\big(1-P\big), \\ \big(1-R\big)^{-1}\big(1-r\big) &= q^{-1}Q, \\ \big(1-R\big)^{-1}r &= \big(1-p\big)^{-1}\big(1-Q\big). \end{aligned}$$

An application: the certainty of a diagnosis

The conditional probability P is thought of as the probability with which the diagnosis of D is confirmed upon a positive T, while Q measures the probability with which the diagnosis of D ought to be ruled out upon a negative T. We can also state the latter probability as the probability with which the diagnosis of $\bar{D}$ is confirmed upon a positive $\bar{T}$. For a particular disease D, the incidence $r = r(D)$ is fixed, and the first equality of equalities (3) can be viewed as an equation, with a constant left hand side, satisfied by P as a function of the variables p and q which, in turn, depend upon the choice of the test T.

For any fixed $r < 1$ and any fixed $p > 0$, P^{-1} approaches 1, and so must P, as q merely approaches 1. In other words, the degree of certainty with which the diagnosis of D is made upon a positive T is fully influenced by q for any positive p. The latter affirmation fails to hold true if we swap p with q. That is, if r and q are fixed, with $rq < 1$, then the inequality

$$\big(r^{-1}-1\big)\big(1-q\big)\le P^{-1}-1$$

implies that P^{-1} is bounded away from 1 and so must P be, regardless of the value of p. So we cannot as fully influence P by merely varying p while fixing q, since P need not approaches 1 if p merely does.

Conclusion: a formal viewpoint

Let d and $\bar{d}$, t and $\bar{t}$ be two dual pairs, and let (d,t) be an ordered pair. Let G denote the group of transformations of the pair (d,t) generated by the three transformations, which we denote by $d \leftrightarrow t$, $d \leftrightarrow \bar{d}$ and $t \leftrightarrow \bar{t}$, induced via swapping d with t, d with $\bar{d}$ and t with $\bar{t}$, respectively.

In the preceding section, we have obtained a correspondence between the ordered pair $(D,\ T)$ with the 10-tuple $\big(p_0,\ p_1,\ p_2,\ p_3,\ p,\ q,\ r,\ P,\ Q,\ R\big)$. Subjecting $(D,\ T)$ to the action of G would yield eight possible ordered pairs. These are

$$(D,\ T),\ (T,\ \bar{D}),\ (\bar{D},\ \bar{T}),\ (\bar{T},\ D),\ (D,\ \bar{T}),\ (\bar{T},\ \bar{D}),\ (\bar{D},\ T),\ (T,\ D).$$

The group G is isomorphic with the eight-element dihedral group, that is, the group of symmetries of the square. Two transformations would suffice for generating G. We fix two such transformations and denote them with g and h. Say g is the transformation obtained via composing the transformation $d \leftrightarrow \bar{d}$ with the transformation $d \leftrightarrow t$, and h is the transformation $d \leftrightarrow t$ itself. With this choice, g^4 and h^2 coincide with the identity transformation of G. There are 8 10-tuples corresponding to the 8 ordered pairs, two of which correspond to the two generating transformations. These are

$$\begin{aligned} g:\ & \big(p_1,\ p_2,\ p_3,\ p_0,\ 1-P,\ 1-Q,\ R,\ 1-q,\ 1-p,\ 1-r\big), \\ h:\ & \big(p_0,\ p_3,\ p_2,\ p_1,\quad P,\quad Q,\ R,\quad p,\quad q,\quad r\big). \end{aligned}$$

Inequality (1) and equality (2) are invariant under the action of the group G. The 4 equalities (4) can be obtained from their 4 corresponding equalities (3) via applying the transformation h. Moreover, either the first two or the last two, of either of the four equalities (3) or (4), can be obtained from each other via applying the transformation

$$g^2:\ \big(p_2,\ p_3,\ p_0,\ p_1,\ q,\ p,\ 1-r,\ Q,\ P,\ 1-R\big).$$

Alternatively, we may obtain the first two of equalities (3) along with the first two of equalities (4) via applying the transformation g to any of the four. And we may obtain the last two of equalities (3) along with the last two of equalities (4) via applying the same transformation g to any of the latter four. Finally, each of the four equalities involving, on the left hand side, both r and R, generates the rest via applying the transformation g.

Note that we have not specified the order in which we composed the transformation $d \leftrightarrow \bar{d}$ with the transformation $d \leftrightarrow t$ in order to produce g. There are two choices here inevitably yielding either g or its inverse g^{-1}. We would have come up with the same two choices had we opted to compose the transformation $t \leftrightarrow \bar{t}$ with the transformation $d \leftrightarrow t$ in one of the two different orders. Each of the transformations $d \leftrightarrow \bar{d}$ and $t \leftrightarrow \bar{t}$ is being conjugate to the other via the transformation $d \leftrightarrow t$. We could have also used either one of them, $d \leftrightarrow \bar{d}$ or $t \leftrightarrow \bar{t}$, to generate, along with $d \leftrightarrow t$, the group G. The latter choice of generators would correspond to generating the dihedral group with two reflections, one of which is about an axis, through the center of the square, parallel to one of its sides and and the other is about a diagonal. Indeed, it is a non-commutative group and the product of its two latter generators does yield an element of order 4. Actually, the symmetry group, of any regualr polygon, not only of a square, routinely generated by a rotation and a reflection, in standard algebra textbooks, can be generated by two reflections alone. A simple, yet too frequently forgotten, a choice for generating the dihedral group by two elements of order 2.

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